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The Franklin Institute
Devoted to Science and the Mechanic Arts

Vol. 247 JANUARY, 1949 No. 1

AMERICAN INVENTORS' DEBT TO THE INSTITUTE.

BY

EVELYN S. PANIAGUA.[1]

One of The Franklin Institute's many claims to fame is its Library's unusually complete collection of U. S. and foreign patent specifications, dating back almost to the inception of The Institute. This collection of information has been of incalculable aid to American inventors and engineers, in enabling them to avoid duplication of effort by seeing what has already been done.

In retrospect it would seem that the very moment The Institute made known its intention to collect information on patent specifications, its laudable purpose would be recognized, and the Government would hasten to offer its cooperation. However, an examination of the early records [2] shows quite the opposite to be true; in fact, the head of the U. S. Patent Office, Dr. William Thornton, proved to be very suspicious of the newly organized Institute. He denied the original request for copies of the specifications and blocked subsequent efforts to obtain them. The record of The Institute's struggle to be recognized reflects great credit on the courage and initiative of the early officers.

The Institute was organized on February 5, 1824. On March 6 of the same year, the Hon. Peter A. Browne, corresponding secretary, addressed the following letter to "Dr. William Thornton, Master of the Patent Office, Washington City":

"Sir,

"A number of mechanics and others of the City have associated themselves together for the purpose of diffusing scientific and useful information among that valuable class of the community and particularly among the rising generation.

[1] Formerly with the Public Relations Department, The Franklin Institute.

[2] Letterbook of the corresponding secretary of The Franklin Institute of Pennsylvania, February, 1824—September, 1826.

The report of the corresponding secretary of The Franklin Institute of the State of Pennsylvania on the subject of the application for copies of the specifications of patents. This covers the matter up to May 13, 1825, with some material not in the Letterbook.

(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the JOURNAL.)

"Their plan embraces an examination into and fair trials of all new inventions. It would therefore be highly useful to them to obtain the earliest and most accurate information relative to all patents taken out in Europe and this Country. With this object in view I have it in charge from the Managers to enquire from you the most favourable terms upon which we can obtain copies of all patents hereafter issued out of your office.

"We do not want certified copies, but the most informal ones that can be taken, provided they are correct. As our object is intimately connected with the public good we have flattered ourselves that our infant institution, whose funds are yet low, will be liberally treated by a gentleman of your known patriotism and public spirit and that you will furnish us with the required information upon as reasonable terms as possible."

This request was answered a few days later by a letter dated March 10, 1824, from Dr. Thornton, saying:

"Sir

"I have just received your Letter relative to the copies of Patents, or rather the specifications of Invention.

"The Law allows copies of Inventions, at the rate of twenty cents for every hundred words. . . .

"I have however considered myself in honor bound to protect the Inventors as much as I can, by allowing (unless in Law Suits where required legally,) no copies of their Inventions which could be used in secret, until their Patents have expired; but any one is at liberty to have copies of those which have expired, & any that have not expired if the consent of the patentee be first obtained: & then a Person might be engaged to copy them at a more reasonable price than the Law allows. . . . Any facilities that depend on me will be granted with the utmost pleasure."

Although Browne was convinced of the unsoundness of Dr. Thornton's position, he desired "to avoid entangling the Institute in the first year of its existence, in a controversy with an officer of the federal government" and "he determined to suspend the prosecution of their rights for a time." [3]

On December 15, 1824, Browne sent out a printed broadside to prospective members, in which he reported on the progress of different projects such as the lectures, exhibitions, and other activities. With regard to the attempt to obtain copies of the patent specifications, he stated, "It will at once be perceived that it is important that the Members and Managers of this Institution, in order to qualify them to serve on committees to examine new inventions, should have an early acquaintance with all new, patented inventions; an effort was therefore made to obtain from the patent office at Washington City, copies of all specifications thereafter issued. I regret to say that owing to an opinion of Dr. Thornton's, (which is believed to be erroneous) that he is under an *honorary* obligation to the patentees to withhold the copies until after the expiration of the term of exclusive privilege, the Institution have not been able to obtain them. Congress have not yet been applied to on this subject, because it is hoped that Dr. Thornton, upon

[3] Report, p. 2.

more mature reflection, and after advising with the Secretary of State, will render that measure unnecessary."

While matters remained in this state, Browne was shown a letter from William P. Elliott, a clerk in the Patent Office, to Edward Clark of Philadelphia, in which Elliott offered to supply, for publication purposes, copies of patents at eight cents per hundred words. He further informed Clark "that Mr. Adams allowed him to make such copies, and that he was in the practice of furnishing them to a Mr. Little of Washington who publishes a similar journal." [3] Browne, in his report to the Board of Managers, commented: "The apparent partiality and injustice of the proceedings at the patent office exhibited by these two statements struck your corresponding secretary at the very first reading of this last letter but he determined not precipitately to condemn a public official but to furnish him an opportunity for explanation or retracting provided a mistake had been made." [3]

Browne accordingly wrote the following letter to Dr. Thornton, dated January 7, 1825:

"Sir,

"I have it in charge from 'The Franklin Institute of the State of Pennsylvania for the promotion of the Mechanic Arts' to enquire of you whether you still adhere to the opinion formerly communicated to me that you are under an 'honorary obligation' to withhold from us the copies of specifications of patents, until after the expiration of the term of exclusive privilege.

"The occasion of troubling you again on this subject is that we entertain a hope that upon more mature reflection and especially after consulting with the Secretary of State you will have found reason to change that opinion."

Failing to receive an answer, Browne became incensed. Under date of January 16, he sent a stinging note to the head of the Patent Office, in which he inquired whether Thornton was "under an honorary obligation not to answer a gentleman's communication on public business." Looking back on the recorded correspondence, some historians might be inclined to regard this action as hasty and ill-considered, even though Browne had legitimate reasons for being annoyed, other than the delay. He must have known that the tone of his letter would not induce the master of the patent office to be more cooperative.

Dr. Thornton's tardy reply, dated January 29, 1825, contained an elaborate explanation of why he had not written sooner. It seems that Browne had not prepaid the postage on his letters, and as Dr. Thornton did not have the franking privilege, they had been sent first to the office of the Secretary of State from whence they were forwarded to the Patent Office. This reply reiterated his refusal to grant the copies requested.

In the meantime, Browne had tired of waiting and on January 29, 1825, had written directly to the then Secretary of State, John Quincy Adams, enclosing copies of all his correspondence with Thornton, and requesting that Adams take a hand in the matter.

"I flatter myself," he wrote, "that you will entertain a different opinion of the

law and . . . that you will take a pleasure in cooperating with [The Institute's] members."

This letter was never answered. In March of 1825, Browne again took up the cudgels in behalf of The Institute, addressing one letter to Henry Clay, the new Secretary of State, and another to Adams, who by that time had assumed the office of President of the United States. After reviewing the circumstances in connection with the case, including the fact that Adams, when Secretary of State, had not deigned to answer his letter, he appealed to the President for intervention in behalf of *The Institute*. He ended his letter to Adams by saying,

"I feel assured that the Chief Magistrate of the United States will not suffer the acts of Congress to be set at naught and the privileges of a free people to be trampled upon by a public officer under his control."

The next setback The Institute received would have been enough to discourage almost anyone else permanently from trying to obtain copies of the patent specifications. On April 10, 1825, Henry Clay sent a copy of an opinion on the matter rendered by William Wirt, the Attorney General. This opinion said, in part:

"The proviso considered with reference to the enactment which it was made to qualify, and with reference also to the essential object of the whole law might well receive the strict construction which the superintendent of the Patent Office has been disposed to place upon it, a construction which would limit the right to demand copies to the Patentee and those who should apply by his authority for such copies. The words, however, are susceptible without violence of a larger construction and they have been accordingly construed to extend to persons who have been sued for a violation of the Patent Right: The law itself expressly authorized such persons to defend themselves . . . [and] it seems but fair that the persons so sued should be furnished with copies of such specifications. To this construction so far as it concerns defendant in suits on Patent, I have heretofore given my official sanction, by an opinion which I presume is on file in the Department of State. But while I should hold it proper to furnish copies for a purpose thus expressly authorized by the law, I should [hold] it equally proper to refuse them not only where there was reason to suspect or even to fear that the purpose might be fraudulent towards the Patentee, but where without such fraudulent purpose, the disclosure might be injurious to the fair interests of the Patentee, as in the case of Patent Medicines, put by the Superintendent."

To continue the struggle in the face of such staggering odds shows that the early Institute members were possessed not only of great courage but also of a profound conviction that they were in the right. They lost no time in enlisting the help of two of the nation's most eminent lawyers, Horace Binney and John Sergeant, who, after reviewing all the facts, rendered an opinion, dated May 10, 1825, which clearly showed the fallacies in Attorney Wirt's line of reasoning. Binney and Sergeant said, in part:

"The recognition of a universal right to copies of such specifications, is we think matter of necessary implication from the Proviso to the 11th Section. It admits that persons may *require* such copies . . . it implies that they may obtain them . . . and it interposes no condition but the payment of a certain fee. That some persons have the right, cannot be questioned, what is there in the *terms* of the Act, or in its

reason or spirit, to confine the right to certain persons, or to make any officer of the government a *judge or tribunal* for ascertaining who those persons are?

"The *terms* of the act are clearly without limitation. The *person obtaining* a copy, is so far as the language of the act is concerned, any person who requires the copy and pays the fee; for although the former part of the section speaks of the inventor, yet, independent of the fact that he possesses his description or specification annexed to or contained in the Patent which is given to him for his thirty dollars, and therefore is not likely to require a particular provision for another, the language of the proviso does not speak of the *inventor*, but of the person obtaining . . . and this change of language shows that some person not the inventor was intended.

"The *reason and spirit* of the law are equally remote from confining the right to certain persons. We apprehend the design of these acts is to give a monopoly for fourteen years, upon the condition of an immediate publication of the invention. The fullest disclosure is required as a condition precedent to the issuing of a patent; its fidelity in all respects is essential to the enjoyment of the patent; and there is not only no stipulation in the act for secrecy or concealment, but the contrary is inferrible from the public office in which all the papers are deposited or recorded, and is in our judgment essential to the due execution of the law, and therefore in coincidence with the intention of Congress. We are not aware to whom the right of copies can be confined, except either to the Patentee or his assigns . . . persons in actual litigation with him or them, or such persons as the Secretary of State shall in his discretion approve. . . . If not so limited, the right must exist in all persons without discrimination.

"A right in the Secretary of State or of any one else to discriminate between the applicants, is no where to be found in the expressions of the law, and it is contrary to the genius of our government and laws, that the discretion of giving or withholding a copy should be asserted by the keeper of public records without an express grant by the legislature. The Secretary has no means of exercising such a discretion soundly, . . . no right to examine the applicant on oath, as to his object in attaining a copy . . . no right to restrain him in the use of it, after it is granted. In regard to these papers, and to their exemplifications, he is not a judicial, but a ministerial officer.

"The right of every one to a copy flows we apprehend, from the patent being matter of public records, and public concern from the papers being deposited in a public office, and from the fees of copying them being prescribed by law without any restraint by the legislature upon the right of requiring a copy."

American technological progress might never have reached its present-day heights if the truth of the following statement had not been recognized:

"It is not the interest of a particular individual merely that is to be regarded in the interpretation of this law, but also that of the public at large. The security of the Patentee is certainly one of the objects of the Act, but his security we apprehend does not lie in the concealment of his specification, nor was it the intention of Congress to promote it in this manner, but it is guarded by the treble damages awarded to him, and by these only; and as well for the purposes of scientific improvement, as for the protection of the public against the frauds of Patentees, it was intended by Congress, and is we conceive essential to the due execution of the law, that specifications should be open and accessible to all persons without discrimination."

So great was the esteem in which Henry Clay held the opinion of Binney and Sergeant that he reconsidered the entire matter in the light of their interpretation. Finally, on September 12, 1825, he despatched the following note to Peter Browne:

"I have availed myself of the earliest opportunity of which circumstances would admit . . . to examine the law in regard to the right claimed by you in behalf of The Franklin Institute, to obtain copies from this office of the specifications accompanying unexpired Patents. The result is a belief that I am authorized to direct

the copies required to be furnished on application. They will accordingly be given whenever they shall be applied for by you in behalf of The Institute, with no other condition than that required by law of the payment, at the rate of twenty cents, for every copy sheet of one hundred words."

On September 26, 1825, the Institute availed itself of its recently recognized right by requesting Dr. Thornton to furnish "a copy of the specification of a patent taken out by Messrs. Charles Cooper and Abraham Justin . . . for an improvement in making iron hoops for the wheels of carriages."

From that time until the present, the Institute has played a consistently important role in encouraging American inventiveness.

Science and industry of the United States would seem to owe a real debt to The Franklin Institute for its perseverance in obtaining this important decision. The free interchange of knowledge thus made possible has undoubtedly hastened our technical leadership.

THE PROBLEM OF ECLIPSING BINARIES.

BY

ISRAEL MONROE LEVITT, Ph.D.[1]

The history of eclipsing binaries began more than a thousand years ago when the star Algol, the second brightest star in the Constellation of Perseus, was seen to undergo regular fluctuations in brightness. Toward the end of the Seventeenth Century this star was an outstanding anomaly, for no one knew nor could guess the cause of the change in brightness. Almost 100 years later John Goodricke, in 1783, communicated to the Royal Society his observations and a conjecture as to the cause of the variation. As is rarely the case in astronomy, a young, unknown amateur, deaf mute Goodricke had discovered the real reason for the variation in brightness, and to this day this is the only type of variable star for which the explanation is complete and theoretically satisfactory. He concluded rightly that the variation was caused by the presence of a second star, which, passing in front of the bright primary, produced a diminution of light in the system. Here was the first indication that there were in the sky such stars as eclipsing binaries.

Almost 100 years later the same star was the subject of an investigation by Professor E. C. Pickering (1) [2] who, on the basis of the light curve, showed that a solution for the elements of the system was possible under certain conditions which were at the time assumed to be reasonable. One wonders just what the state of the subject would have been had there been no star like Algol or had Algol been just a little bit fainter.

After the initial attack by Pickering others like Hartwig, Tisserand, A. W. Roberts and Meyers continued the investigation and produced results which finally culminated in the monumental work of Professor Henry Norris Russell, published in 1912 (2).

Until the works of these men were available the problem was thought to be of such difficulty that a general solution was impossible. For instance, Bauschinger (3), in his exhaustive treatise on the determination of orbits in astronomy, stated, "The relationship between the ratios of size, shape and brightnesses of stars and of the elements of the elliptic orbit on one hand and the light curve on the other is so complicated that one can scarcely formulate a general theory, but one must adjust the solution for each individual case to the respective condi-

[1] Director, Fels Planetarium, The Franklin Institute; and Astronomer, Cook Observatory, University of Pennsylvania, Philadelphia, Penna.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

tions."[3] The difficulty of this phase of double star astronomy was recognized and is still recognized, and even today, more than 30 years after the initial attack by Russell, the complete solution is still nowhere in sight.

When the problem is considered, it is remarkable that from such meager information as that furnished by light curves the elements of a binary system can be established. And when it is established that the initial work began on light curves which were visual or photographic, the magnitude of the work of the pioneers in the field is appreciated.

The astronomer begins the solution of the problem by obtaining a light curve. The light curve establishes the variation of the total light of the system seen as one star. From it one can accurately determine the period of the variation in the brightness of the pair. The curve discloses further the fractional loss of light which the system undergoes during eclipse, the effects of ellipticity and reflection over the entire cycle—this effect is most apparent between eclipses—and the anomalies which may be found in the system, such as, rotation of the line of apsides, high eccentricity of the orbit, effects of a third body, periastron effects, etc.

To discuss the eclipse and arrive at conclusions concerning the star system it is necessary to remove from the light curve all the effects of ellipticity and reflection. This essentially adjusts the light curve so the intensity between minima is a constant. The removal process, called rectification, is not, even after 40 or more years of intensive research, too well defined and requires sound judgment on the part of the computer for proper application. Usually the light curve between eclipses (here is where the computer must exercise sound judgment) is subjected to a rather simple Fourier analysis involving only functions of θ and 2θ. With the accuracy of the observations attained today this is sufficient to get the necessary accuracy from the observations. Conceivably, when observation becomes more accurate, terms of higher order may also be used.

Rectification also discloses the presence of any asymmetry in the light curve. Once this asymmetry is obtained the curve is rectified for it even though no explanation can be assigned to its presence. Some observers have devised weird star shapes and systems to account for the asymmetry but too little is known about the stars themselves to pay serious attention to these claims.

In the usual case light curves are obtained directly in magnitudes, and, to facilitate the work, these magnitudes are changed to light units and the curve is operated on in this changed system.

[3] Translated from the original, which reads: "Der Zusammenhang zwischen den Grossen, Formen- und Helligkeitsverhaltnissen der Korper und der elliptischen Bahn einerseits und der Lichtkurve anderseits ist aber ein so komplizierter, dass man eine allgemeine Theorie wohl kaum aufstellen kann, sondern die Losung von Fall zu Fall den vorliegenden Verhaltnissen anpassen muss."

In determining the elements from the rectified light curve some basic assumptions must be made. The light curve is assumed to be plotted in light units, and times of minima and the period of variation are assumed known. As this article is a survey article, the mechanics of the above steps are not important though the results are of paramount importance in the solution of the problem.

Two sets of symbols will have to be used. One set will define the complete specifications of the system under consideration, and these might be called the geometric elements of the system. For the purpose of this paper the notation used will be that of Irwin (4).

1. Semi-major axis of the relative orbit ..... a
2. Eccentricity ..... e
3. Longitude of periastron ..... ω
4. Longitude of ascending node ..... Ω
5. Orbital inclination with respect to plane of sky i
6. Period ..... P
7. Epoch of principal conjunction ..... l_0
8. Radius of smaller star ..... r_2
9. Radius of larger star ..... r_1
10. Light of smaller star ..... L_2
11. Light of larger star ..... L_1
12. Degree of darkening of smaller star ..... x_2
13. Degree of darkening of larger star ..... x_1
14. Elements pertaining to shapes of the stars ..... $a:b:c$
15. Constant representing the reflection effect ..... A_1

The second set of symbols is germane to the solution of the problem and involves the parameters of the solution. These, whose meanings will become apparent as the development proceeds, are:

1. Normalized fractional loss of light of the star undergoing eclipse ..... α
2. Fractional loss of light of the star undergoing eclipse ..... f
3. Geometric depth of eclipse ..... p
4. Ratio of the radii of the two stars and always taken as r_2/r_1 ..... k
5. Coefficient of limb darkening x
6. Mean surface brightness ..... J
7. Fractional light of system during eclipse l
8. Value of l at mid minima ..... λ
9. Phase angle of companion in system ..... θ

Of the elements given in the first set of symbols there are many which must remain unknown now and some which can never be determined. The absolute value of a can be obtained only from spectroscopic information which yields displacements of the lines and a knowledge of the parallax. In the absence of this determination the value of a is set arbitrarily as 1 and the radii of the stars determined in the terms of this.

In the case of e, the eccentricity, it is well established that among eclipsing binary systems orbital eccentricities are so small that a circular

orbit is considered a safe approximation. With a circular orbit

$$e \cos \omega = e \sin \omega = 0$$

so effectively not only e, but ω, the longitude of periastron, is eliminated as a parameter to be determined.

Since there is virtually no hope of resolving an eclipsing binary visually, Ω, the longitude of the ascending node is undeterminable.

Of the remaining parameters the orbital period, P, is known with an accuracy far exceeding that obtainable for any of the other elements and likewise t_0, the epoch of primary minimum, is directly determinable from the observations and is independent of other elements.

Unless the parallax of the system is known, the absolute values of L_1 and L_2 cannot be determined. Consequently, in analyzing the problem the combined light of the system, that is, $L_1 + L_2$ is taken as the unit of light and the brightness of the system at any epoch is stated in terms of this.

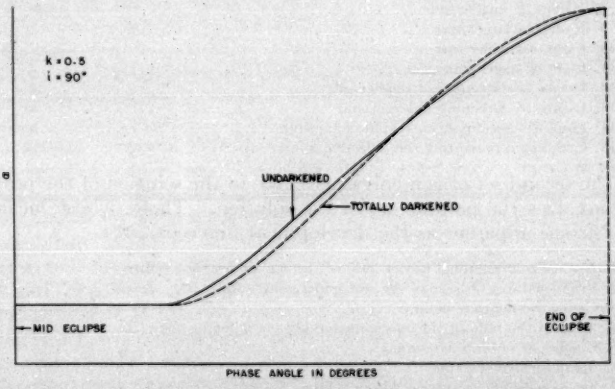

FIG. 1. Normalized fractional loss of light of star being totally eclipsed.

In the case of the coefficients of limb darkening, x_1 and x_2, these vary with spectral type and the wave length of observation. When the spectral types are unknown Russell (5) says, "It is assumed that the limb darkening is the same for both components and has a value specified in advance." Russell (6) further not only gives a table with the limb darkening as a function of the spectral type, that is, the temperature, but also gives the value of the gravity darkening for the given condition. Even though this table is in error it can be used as a starting point and this effectively eliminates the limb darkening as a parameter to be determined.

Though limb darkening is extremely important in the problem of

eclipsing binaries this paper is too limited in scope to consider the subject in detail. However, it can be demonstrated that limb darkening does affect the shape of the light curve. Figure 1 illustrates the case of a total eclipse both totally darkened, $x = 1$ and undarkened, $x = 0$.

It is apparent that the light begins to fall off more slowly in the case of the darkened star. Inasmuch as there is complete limb darkening, the eclipsing star cuts off less light from the eclipsed star at the beginning of the eclipse simply because there is less light to cut off. Somewhere between first and second contact the fractional losses of light of the darkened and undarkened stars are equal and then the loss becomes greater for the completely darkened star.

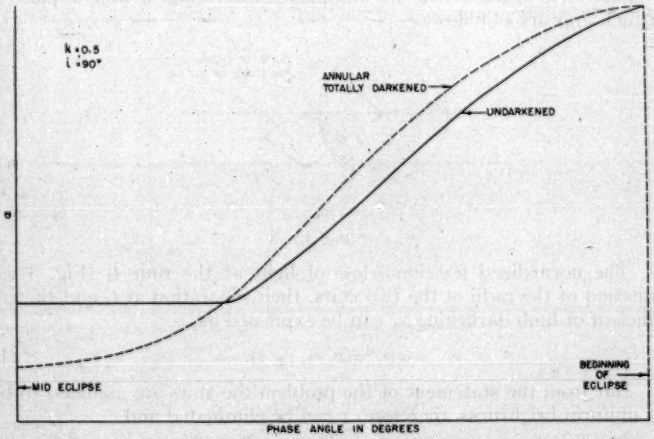

FIG. 2. Normalized fractional loss of light of star undergoing annular eclipse.

Figure 2 illustrates the case of an annular eclipse and what was said of Fig. 1 holds for Fig. 2 until the point of second contact. As the eclipsing star is smaller than the eclipsed star an annulus of light is always present and the light from the eclipsed star reaches a minimum at mid eclipse. This feature makes it difficult to differentiate between a darkened annular eclipse and a partial eclipse. For this reason the secondary minimum must be available for the solution.

The remaining parameters are r_1, r_2, i and L_2 (as $L_1 = 1 - L_2$ the determination of one automatically yields the other). These are the parameters which result from the solution of the eclipsing binary problem.

Russell (2, p. 317) in his early analysis of the problem makes several assumptions. He says, "Two spherical stars, appearing as uniformly il-

luminated discs, and revolving around their common center of gravity in circular orbits, mutually eclipse one another. It is required to find the relative dimensions and brightnesses of the two stars, and the inclination of the orbit, from the observed light curve." For the simple solution it must be noted further that the minima are assumed to be symmetrical and of equal duration and are separated by half the orbital period. The deeper minimum corresponds to the eclipse of the star of greater surface brightness, as is customary in the interpretation of the problem dealing with uniformly bright stars.

In the foregoing pages enough has been given to see the almost unbelievable complexity of the problem which on the surface might appear simple. The parameters are complexly intermingled and a plan of attack appears as follows.

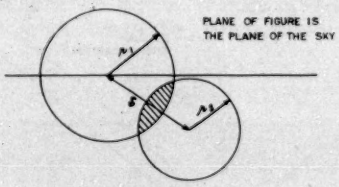

FIG. 3.

The normalized fractional loss of light at the time t_1 (Fig. 3); a function of the radii of the two stars, their separation at t_1 and the coefficient of limb darkening x; can be expressed as

$$\alpha \equiv \alpha(x, r_1, r_2, \delta). \tag{1}$$

But from the statement of the problem the stars are assumed to be of uniform brightness, therefore, x can be eliminated and

$$\alpha^U \equiv \alpha^U(r_1, r_2, \delta) \tag{2}$$

where α^U indicates the normalized fractional loss of light of uniformly illuminated stars and α and f are related where $f = q\alpha$ and *q is the fractional loss of light at internal contact.*

If the independent variables in (2) are divided by r_1,

$$\alpha^U \equiv \alpha^U\left(\frac{r_2}{r_1}, \frac{\delta}{r_1}\right) \tag{3}$$

which involves only two variables.

If now k is introduced

$$k \equiv \frac{r_2}{r_1} \tag{4}$$

where r_2 is the radius of small star, r_1 is the radius of large star, and $0 \leq k \leq 1$.

The geometrical depth of eclipse p can be introduced as

$$p \equiv \frac{\delta - r_1}{r_2} \tag{5}$$

and it can be seen that p can vary in the following limits from first contact to second contact.

$$-1 \leq p \leq +1.$$

Values of $p < -1$ can occur in the annular phases of eclipses. From the above it is clear that in the case of a total eclipse .

$$\alpha^U \equiv \alpha^U(k, p). \tag{6}$$

An important relation involving δ, k, and p can be determined as shown in Fig. 4, where θ is a function of the period and is equal to

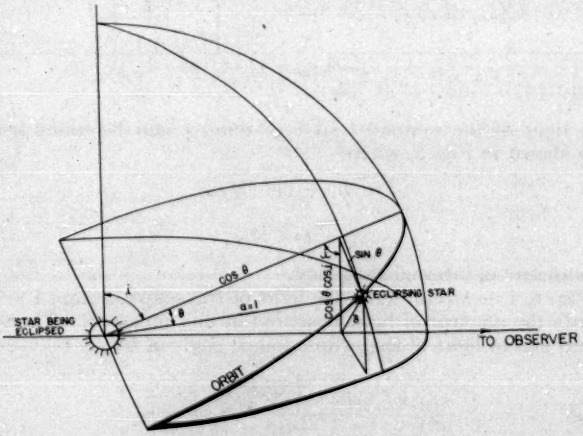

FIG. 4.

$(2\pi/P)(t - t_0)$; t equals time of any phase of eclipse; and i = inclination. From the geometry of Fig. 4,

$$\delta^2 = a^2 \sin^2 \theta + a^2 \cos^2 \theta \cos^2 i. \tag{7}$$

For convenience let a, the radius of the orbit, equal 1, then

$$\delta^2 = \sin^2 \theta + \cos^2 \theta \cos^2 i.$$

Now

$$p = \frac{\delta - r_1}{r_2}$$

and

$$p = \frac{\frac{\delta}{r_1} - 1}{k}. \tag{8}$$

Therefore

$$\delta = r_1 + r_2 p = r_1(1 + kp), \tag{9}$$

$$\sin^2\theta + \cos^2\theta \cos^2 i = \delta^2 = r_1^2(1 + kp)^2 \tag{10}$$

and

$$\cos^2 i + \sin^2 i \sin^2\theta = \delta^2.$$

Since α or f are known from the light curve for a total eclipse we have the relation among the three variables i, r_1, and k. *This is the first fundamental equation of the problem.*

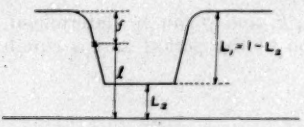

FIG. 5.

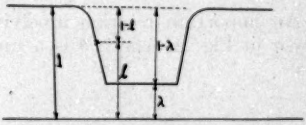

FIG. 6.

The light of the system, l, at any time, t, can be found from the relation shown in Fig. 5, where

$$l = L_1 + (1 - f)L_2$$

or

$$l = 1 - fL_2$$

at the moment of internal tangency.

In Fig. 6, $1 - \lambda$ represents the light of the eclipsed star; $1 - l$ then represents the amount of light obscured at any moment. Then α, the fractional loss of light of the eclipsed star, is given by

$$\alpha = \frac{1 - l}{1 - \lambda} \tag{11}$$

and at the time of maximum obscuration

$$\alpha = \frac{1 - l}{1 - \lambda}\alpha_0$$

where $\alpha_0 = \alpha(k, p_0)$ denotes the maximum obscuration.

Now consider any three points on the light curve shown in Fig. 7.

FIG. 7

By Eq. 10 we can write

$$\begin{aligned} \sin^2\theta_1 \sin^2 i + \cos^2 i - r_1^2(1+kp_1)^2 &= 0 \\ \sin^2\theta_2 \sin^2 i + \cos^2 i - r_1^2(1+kp_2)^2 &= 0 \\ \sin^2\theta_3 \sin^2 i + \cos^2 i - r_1^2(1+kp_3)^2 &= 0 \end{aligned} \tag{12}$$

$$\Delta = \begin{vmatrix} \sin^2\theta_1 & 1 & -(1+kp_1)^2 \\ \sin^2\theta_2 & 1 & -(1+kp_2)^2 \\ \sin^2\theta_3 & 1 & -(1+kp_3)^2 \end{vmatrix} = \begin{vmatrix} \sin^2\theta_1 & (1+kp_1)^2 & 1 \\ \sin^2\theta_2 & (1+kp_2)^2 & 1 \\ \sin^2\theta_3 & (1+kp_3)^2 & 1 \end{vmatrix}. \tag{13}$$

If these linear homogeneous equations are consistent

$$\Delta = 0. \tag{14}$$

Then

$$\sin^2\theta_1[(1+kp_2)^2-(1+kp_3)^2]-\sin^2\theta_2[(1+kp_1)^2-(1+kp_3)^2] +\sin^2\theta_3[(1+kp_1)^2-(1+kp_2)^2]=0$$

$$\sin^2\theta_1[(1+kp_2)^2-(1+kp_3)^2] =\sin^2\theta_2[(1+kp_1)^2+(1+kp_2)^2-(1+kp_2)^2-(1+kp_3)^2] -\sin^2\theta_3[(1+kp_1)^2-(1+kp_2)^2]$$

$$\sin^2\theta_1[(1+kp_2)^2-(1+kp_3)^2]=\sin^2\theta_2[(1+kp_2)^2-(1+kp_3)^2] + (\sin^2\theta_2-\sin^2\theta_3)[(1+kp_1)^2-(1+kp_2)^2], \tag{15}$$

$$\sin^2\theta_1=\sin^2\theta_2+(\sin^2\theta_2-\sin^2\theta_3)\,\frac{(1+kp_1)^2-(1+kp_2)^2}{(1+kp_2)^2-(1+kp_3)^2}, \tag{16}$$

$$\sin^2\theta_1=\sin^2\theta_2+(\sin^2\theta_2-\sin^2\theta_3)\,\frac{2(p_1+p_2)+k(p_1^2-p_2^2)}{2(p_2-p_3)+k(p_2^2-p_3^2)}. \tag{17}$$

Let

$$A = \sin^2\theta_2, \tag{18}$$

$$B = \sin^2\theta_2 - \sin^2\theta_3 \tag{19}$$

and

$$\psi = \frac{2(p_1-p_2)+k(p_1^2-p_2^2)}{2(p_2-p_3)+k(p_2^2-p_3^2)}. \tag{20}$$

Then

$$\sin^2\theta_1 = A + B\psi,$$

since

$$\begin{aligned} p &\equiv p(k,\alpha) \\ \psi &\equiv \psi(k,\alpha) \\ \alpha &\equiv \alpha(k,p). \end{aligned}$$

If the θ_2, p_2, α_2 are taken arbitrarily when $\alpha_2 = 0.6$ and θ_3, p_3, α_3 taken when $\alpha_3 = 0.9$, tables of $\psi(k,\alpha)$ can be tabulated as functions of k and α.

It is seen that a knowledge of ψ for any α leads to a determination of k. Once having obtained k, we can go back to Eq. 10 and find r_1 and i.

These k will be in error, of course, due to the inaccuracies in computing ψ from the observations. But it is also known that k must be

the same for all α's since the radii of the two stars are constant. The obvious approach to the problem of getting a constant value of k for all solutions is to adjust ψ in some way.

The constants, A and B, are only known with observational accuracy, and it is convenient to adjust their values to get a better fit to k in the neighborhood of $\alpha = 0.6$ and $\alpha = 0.9$. In practice, A is adjusted first to get the best fit at $\alpha = 0.6$. The ψ and k are then recomputed. B is adjusted to give a better fit around $\alpha = 0.9$. The ψ and k are then recomputed for this condition. If the k's are not too discordant, the mean k is found with the following weights:

zero wt. for $\alpha = 0.0, 0.6, 0.9, 1.0$

single wt. for $\alpha = 0.1, 0.5, 0.7, 0.8$

double wt. for $\alpha = 0.2, 0.3, 0.4, 0.95, 0.98, 0.99$

When a satisfactory light curve has been computed the remaining elements r and i may be computed.

It can readily be shown that

$$A = (1 + kp_2)^2 r_1^2 \csc^2 i - \cot^2 i, \tag{21}$$

$$B = [(1 + kp_2)^2 - (1 + kp_3)^2] r_1^2 \csc^2 i. \tag{22}$$

Therefore

$$\cot^2 i = \frac{(1 + kp_2)^2 B}{(1 + kp_2)^2 - (1 + kp_3)^2} - A. \tag{23}$$

Also,

$$r_1^2 \csc^2 i = \frac{B}{(1 + kp_2)^2 - (1 + kp_3)^2}. \tag{24}$$

Let

$$\Phi_2(k) = 1 - \frac{(1 + kp_3)^2}{(1 + kp_2)^2}$$

and

$$\Phi_1(k) = \frac{1}{(1 + kp_2)^2 - (1 + kp_3)^2}.$$

These Φ functions are tabulated by Russell (2, p. 337; 7). Then

$$r_1^2 \csc^2 i = \frac{B}{\Phi_1(k)} \tag{25}$$

and

$$\cot^2 i = \frac{B}{\Phi_2(k)} - A. \tag{26}$$

The elements r and i are derived from these.

If $\cot^2 i < 0$ it is negative or imaginary and the solution is physically impossible. The $\cot^2 i$ is then set at zero, which corresponds to a central transit, and A and B are readjusted.

Using this final k compute ψ, ψB, $\sin^2 \theta_1$ and finally t for each α.

Plot the theoretical light curve and get the residuals for each normal or observed point from the plot.

The last thing is to reassemble the elements, k, i, r_1, r_2, L_1 and L_2.

The solution of the problem as worked out by Russell is simple and, because of its very simplicity, has a number of disadvantages, none overcoming its prime advantage of being one of the shortest methods.

For instance, everything depends on A and B which unfortunately are determined initially from a freehand curve and further there is no satisfactory system for weighting the α's.

This development, which can be used to solve the problem in the case of a total eclipse, can be extended to the annular and partial types of eclipses.

In the case of the annular eclipse the determination can be made from the geometry of the problem if the secondary minimum has been observed.

In Fig. 8, if no darkening is assumed, then, for a total eclipse

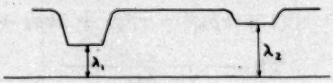

FIG. 8.

$\lambda_1 = L_1$ = light of the larger star

and for an annular eclipse

$\lambda_2 = L_2 + L_1(1 - k^2)$ = light of smaller star plus the rim of larger star

where

λ_1 = fractional light of system at conjunction when large star is in front.
λ_2 = fractional light of system at conjunction when small star is in front.

$$k^2 = \frac{1 - \lambda_2}{\lambda_1} \leqq 1,$$

rewriting:

$$\lambda_1 = \frac{1 - \lambda_2}{k^2}. \tag{29}$$

In an annular eclipse

$$1 - \lambda_2 \leqq k^2$$

in order to satisfy the above equation (note $\lambda_1 \leqq 1$).

This is a criterion for the existence of an annular eclipse and it fails when the minima approach equal depths.

Russell's method lends itself to the solution of partial eclipses though, as will be shown, the determination is not so clearly defined as it is for total or annular eclipses.

Consider Fig. 9 showing the primary minimum of a partial eclipse.

In Fig. 9 we begin with two equations

$$\frac{1-l}{1-\lambda} = n \tag{30}$$

and

$$\alpha = n\alpha_0. \tag{31}$$

At the time of conjunction

$$\begin{aligned} p &= p_0 \\ n &= 1 \\ \theta &= 0 \\ l &= \lambda \\ \alpha_0 &= \text{maximum obscuration} \end{aligned}$$

and Eq. 10 becomes

$$\delta^2 = \sin^2 i \sin^2 \theta + \cos^2 i = r_1^2(1 + kp)^2 \tag{10}$$

where p (geometric depth of eclipse at any phase) becomes, at the moment of maximum eclipse

$$\cos^2 i = r_1^2(1 + kp_0)^2 = r_1^2(1 + 2kp_0 + k^2p_0^2). \tag{32}$$

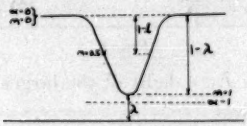

FIG. 9.

If Eq. 32 is subtracted from Eq. 10 we get

$$\sin^2 \theta = r_1^2 \csc i[k^2(p^2 - p_0^2) + 2k(p - p_0)] \tag{33}$$

or

$$\sin^2 \theta(n) = r_1 r_2 \csc i[k(p^2 - p_0^2) + 2(p - p_0)]. \tag{34}$$

If a point on the light curve is picked where n = 0.5, then

$$\sin^2 \theta(0.5) = r_1 r_2 \csc^2 i[k(p_1^2 - p_0^2) + 2(p_1 - p_0)] \tag{35}$$

where p_1 corresponds to n = 0.5, as $p = p(k, \alpha_0, n)$; $p_1 = p(k, \alpha_0, \frac{1}{2})$; and $p_0 = p(k, \alpha_0)$. If we divide Eq. 39 by Eq. 35 we get

$$\chi(k, \alpha_0, n) = \frac{\sin^2 \theta(n)}{\sin^2 \theta(0.5)} = \frac{k(p^2 - p_0^2) + 2(p - p_0)}{k(p_1^2 - p_0^2) + 2(p_1 - p_0)}, \tag{36}$$

which is an equation in three unknowns, only one of which (n) is known from the light curve.

However, χ can be tabulated and Russell in his original papers gives the values of χ for k and α_0 for three points on the curve: n = 0; n = $\frac{1}{4}$; and n = $\frac{3}{4}$.

Now if we let

$$\chi(k, \alpha_0, \tfrac{1}{4}) = C_1^*$$
$$\chi(k, \alpha_0, \mathrm{n}) = C_2$$

the C_1, C_2 curves can be plotted in k, and α_0 coordinates and when the plot is complete we get Fig. 10. And point A, the intersection of the two curves, gives the desired values of k and α_0 for the system. However, although this appears reasonable it has severe limitations. The two curves are nearly functionally dependent and in some cases are almost parallel so they cross at slight angles. The result from a cross-over point so ill-defined gives a large variation in the answers and as a result the k and α_0's are poorly determined. This indeterminacy for uniform stars is even worse than that for darkened stars. Once k and α_0

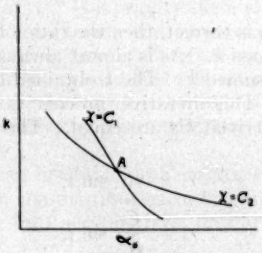

FIG. 10.

are determined, Eq. 10 will yield the remaining geometrical elements.

The difficulties surrounding the determination of k and α_0 mean that the Russell method, so effective in dealing with the problem of a total eclipse, is in some cases completely inadequate to handle the problem of partial eclipses.

Other investigators have attacked the problem of partial eclipses and at least two have had a good bit of success in deriving the elements.

Kopal, for instance, begins at the same starting point as Russell that is, with Eq. 10.

$$\sin^2\theta \sin^2 i + \cos^2 i = r_1^2(1 + kp)^2. \tag{10}$$

At the moment of conjunction

$$\cos^2 i = r_1^2(1 - kp_0)^2. \tag{37}$$

Subtracting Eq. 37 from Eq. 10 gives

$$\sin^2\theta(\mathrm{n}) = (p^2 - p_0^2)C_1 + 2(p - p_0)C_2 \tag{38}$$

* C_1 and C_2 are not related to C_1 and C_2 of the Russell notation.

where

$$p = p(k, \mathrm{n}, \alpha_0)$$

$$p_0 = p(k, \alpha_0)$$

$$C_1 = r_2^2 \csc^2 i \tag{39}$$

$$C_2 = r_1 r_2 \csc^2 i. \tag{40}$$

In the above expressions, θ and n are knowns, while C_1, C_2 and p_0 are unknowns.

Kopal assumes a provisional value of k. Then $p^2 - p_0^2$ and $p - p_0$ can be obtained from p tables with this k.

An equation of condition for each normal (observed) point can be set up with C_1, C_2 as unknowns and these determined by a least squares solution.

If the assumed k was correct, then the ratio of C_1 to C_2 should yield the value of the assumed k. As is almost always the case this ratio of C_1 to C_2 is not the assumed k. The k obtained by C_1/C_2 is used for a new determination. This iterative process is continued until the assumed k and the derived k's are equal. The elements then follow from:

$$r_1 = \sqrt{C_1} \sin i, \tag{41}$$

$$r_1 = \frac{C_2}{\sqrt{C_1}} \sin i, \tag{42}$$

$$\cot i = \frac{p_0 C_1 + C_2}{C_1}. \tag{43}$$

The treatment for the solution of the eclipsing binary problem has so far taken no account of the true characteristics of the stars but has dealt with an idealized case as noted previously.

Even spherical stars do not have a uniform light distribution over the entire surface of the disc. A photograph of the sun immediately discloses the presence of the limb darkening and it is fairly certain that practically all stars exhibit this characteristic. Likewise it is fairly certain that full darkening of the limb is rarely realized in practice. As a result, the astronomer has to deal with stars which have varying degrees of limb darkening and in the final solution must be able to estimate the limb darkening of the stars in the system from the spectral types.

In the early days of the development of the eclipsing binary theory χ and ψ tables were constructed experimentally by dividing the disc of a star radially and annularly into many small areas of equal brightness which were counted to arrive at the fractional loss of light at a certain phase. Today in the elegant treatment given to this problem by Kopal

and others, in the form of more accurate tables, it is possible to solve the problem with considerably more accuracy.

Stars which are members of an eclipsing binary system are not always spherical bodies. Being gaseous they can be deformed readily to attain a state of equilibrium. The actual shape of each star depends upon its size, mass distribution and speed of rotation, as well as upon the separation and mass of both components. However, in practice it may be assumed the stars are generally deformed into triaxial ellipsoids, with the shortest axis being perpendicular to the plane of the orbit and the longest axis being along the line joining the centers of the stars.

Von Zeipel (8) has shown that the outward flux of radiation, that is, the surface brightness, over distorted surfaces will vary as a function of the local gravity. This gravity darkening is superposed on the limb darkening. Kopal (9) says, "This gravity effect should naturally superpose upon ordinary limb darkening and give rise to a complicated system of isophotae on apparent disks of distorted stars. The distortion of components in close eclipsing systems will, therefore, influence their light curves both directly, by varying the area of each star exposed to the observer, and indirectly, by rearranging the distribution of brightness upon it."

However, it is not an insoluble problem, for Kopal has derived expressions which govern the distribution of brightness over the discs of distorted stars, taking into account both limb darkening and gravity darkening.

There are two other quantities whose effect on the light curve must be included for a full solution to the problem. These are the ellipticity and reflection coefficients.

In close binary systems the variation in light not only is present during the times of minima but extends over the entire cycle. As we have seen, the stars are assumed to be triaxial ellipsoids with the long axis in the line joining the centers. As the two stars revolve about their common center of gravity it is apparent that the surface areas of the stars in the line of sight will vary continuously in the course of a revolution.

Also in close binary systems some radiation from each star will fall upon the other star which will be received and reradiated into space. Some of this radiation will find itself in the line of sight and may be considered as superposed on the intrinsic radiation of the star. Here again we have the equivalent of two sources of radiation for each star complexly intermingled and again varying with the phase. This plethora of changes in the light of the eclipsing binary system renders the problem one of the most difficult in the field of astronomy.

The above discussions of the light fluctuations have by no means exhausted the conditions that arise to warp and affect the light coming from a system.

We have assumed circular orbits and in most cases this assumption is a valid one; but the effects of orbital eccentricity become apparent in some cases. In some light curves the times of primary and secondary minima will be unequally spaced and, when plotted against a constant period, will deviate in approximately a sine curve with a period equal to the time of rotation of the line of apsides. Occasionally the secondary minimum will be found "off center." Here again the cause is laid to orbital eccentricity even though at this writing there are very few cases exhibiting this effect.

A light curve may be investigated to disclose a variation in period. Here again there is more than one cause to give this effect. The variation in period may be due to a rotation of the line of apsides and this effect may be appreciable in highly eccentric orbits.

If a variation of period is found and the secondary minimum persists at the half way point, the cause for this variation is very likely the presence of a third body.

There are still other effects such as asymmetry in the light curve, the periastron effect which produces greater distortions in the stars and therefore affects their reflections and still other effects on the light curve which may be minor but nevertheless are still present.

The elements derived from these solutions are satisfactory and valid because the errors within the elements usually are within the errors of the observations. There is a possibility for refining these elements but as a rule the results do not warrant this refinement because essentially the observations do not merit it.

A least squares solution by differential corrections is possible for any set of observations but it will not be applicable if the residuals are large. Essentially a least squares solution is based on a linearization of the differential corrections and assumes the residuals in intensity are small compared to total brightness. In general, the only observations which merit least squares correction are those resulting from photoelectric photometry.

In making these corrections certain functions such as

$$\frac{\partial f}{\partial r_1}, \frac{\partial f}{\partial r_2}, \frac{\partial f}{\partial i}, \frac{\partial f}{\partial x} \text{ and occasionally } \frac{\partial f}{\partial t_0}$$

must be evaluated, where r_1, r_2, i and x are the same quantities met previously in the solution of the problem. Up to a very recent date there was no table of these functions and in any work requiring least square corrections these coefficients had to be computed and set up. The late Dr. A. B. Wyse (10) investigated this approach and gave the first exhaustive treatment of the problem in which not only was k adjusted, as was common by the early methods, but all elements were subject to an appropriate adjustment.

Later Kopal (11) improved on the treatment and there resulted the associated alpha functions which implicitly contained all the correction functions.

Within the past year Irwin (4) has published complete tables of the coefficients for the application of the least squares solution to the problem of eclipsing binaries.

During the past 30 years the accuracy of the observations has improved to the point at which complete mathematical treatment is essential to draw from the observations everything possible. In the old days when visual work was predominant the accuracy attained may have been on the order of ± 0.04 magnitude. The advent of photoelectric photometry has increased this accuracy by a factor of between 10 and 20. Today observations with accuracies of ± 0.003 magnitude are commonplace. Kron in his work on *YZ Cassiopeiæ* has attained the remarkable accuracy of ± 0.001 magnitude. Within the foreseeable future improvements in electronic techniques and improvements in observational techniques, such as simultaneous comparisons of variable and comparison, may improve the observations to the point where the probable error of a normal point for any star may be on the order of ± 0.001 magnitude. With observational skill perfected to the point where the probable errors are minute, the mathematical technique to operate on these observations must of necessity become more and more refined. Kopal's work has already furnished the mathematical tools for analyzing the light curve to allow the elements to be derived with a great deal of refinement. At this very moment it is possible to represent the light variation of any distorted system to the required degree of accuracy. But the reverse process is, in general, a good bit more difficult, and this very difficulty has given rise to the refinements in the theory. Now once more the observational work must progress if the tables already constructed and in use are to be employed efficiently.

It would be interesting to be able to project ourselves into the future and see the observational accuracy attained. Some astronomers, at this moment, doubt that the tremendous increase in accuracy, reached in the past 30 years can be duplicated in the future. Physical handicaps, such as atmospheric disturbances and inherent instabilities in photometers will be the ultimate limiting factors in the astronomer's struggle for greater and greater accuracy, and when we finally reach the moon and begin observing from it with stable electronic equipment, the quantum nature of radiation will effectively act as a brake to the attainment of unlimited accuracy.

An application of the foregoing theory to the solution of the eclipsing Binary, ZZ Cassiopeiae, will be presented in the next issue.

REFERENCES.

(1) E. C. PICKERING, *Proc. Amer. Acad. Arts and Sciences*, Vol. 16, p. 1 (1881).
(2) H. N. RUSSELL, *Ap. J.*, Vol. 35, p. 315 (1912).
(3) J. BAUSCHINGER, "Die Bahnbestemmung der Himmelskorper," Leipzig, 1906, p. 649.
(4) J. B. IRWIN, *Ap. J.*, Vol. 106, p. 380 (1947).
(5) H. N. RUSSELL, *Popular Astronomy*, Vol. 54, p. 169 (1946).
(6) H. N. RUSSELL, *Ap. J.*, Vol. 102, p. 5 (1945).
(7) H. N. RUSSELL, *Ibid.*, Vol. 36, p. 246 (1912).
(8) H. VON ZEIPEL, *M. N.*, Vol. 84, p. 702 (1924).
(9) ZDENEK KOPAL, "Introduction to the Study of Eclipsing Variables," Cambridge, Harvard University Press, 1946, p. 110.
(10) A. B. WYSE, *Lick Bull.*, No. 494 (1936).
(11) ZDENEK KOPAL, *Proc. Amer. Phil. Soc.*, Vol. 86, p. 342 (1943); *Ibid.*, Vol. 88, p. 145 (1944).

V.
19

STRAND VIBRATIONS IN A ROLLER-CHAIN DRIVE.

BY

R. C. BINDER [1] AND G. G. MIZE.[2]

SUMMARY.

Under some operating conditions the roller-chain strand between sprockets may have a marked vibration, whereas under other conditions most of the strand may be straight. This paper presents an analytical study of the tight strand vibration. It is assumed that the strand is equivalent dynamically to a string in tension, and loaded with a mass at each roller center. A relation for natural frequencies is given. An impact or impressed frequency close to one of the natural frequencies may cause a marked vibration of the strand. The results of some tests are presented. There is fair agreement between the experimental results and the theoretical predictions.

One method of connecting two rotating shafts is to use a roller chain and two sprockets. A factor influencing the performance of the drive is the action of the tight chain strand transmitting a force from one sprocket to the other. Observations indicate a variety of chain paths. Under some operating conditions the strand between sprockets may have a marked vibration, whereas under other conditions most of the strand is straight. The following discussion presents a study of the tight strand vibration. Such an analysis may help in correlating experimental data, in explaining some qualitative observations found in practice, and in meeting special cases involving vibration features.

Figure 1 is a diagrammatic sketch of a drive. It is assumed that the sprockets are of the same size, that sprocket pitch exactly equals new chain pitch, and that the driver rotates at constant angular speed. At very low angular speeds of the sprockets, the entire tight strand may behave like the rigid connecting rod of a four-bar linkage. Each sprocket resembles a crank. A roller center in the strand then follows a path consisting of circular arcs each with a radius equal to the pitch radius of the sprockets.

The action at relatively high speeds requires a study of the inertia and elastic properties of the chain strand. There is usually a large number of links in the chain, and each link consists of a number of different parts. The vibration problem is thus very complicated, and it is necessary to make certain assumptions in order to provide a practical, usable calculation. Considering the uncertainty of the loads which may act on a drive at times, a simplified analysis may be the wisest plan.

For the conditions illustrated in Fig. 1, the tight strand is supported at each end by a sprocket tooth, at the rollers A and B. It will be as-

[1] Professor of Mechanical Engineering, Purdue University, Lafayette, Ind.

[2] Chief Engineer, Diamond Chain Co., Inc., Indianapolis, Ind.

sumed that the strand is equivalent dynamically to a string in tension, and loaded with a mass at each roller center. The masses are all equal. There is this question: why not treat the strand as a continuous string? Such a substitution would not take into account the finite and discrete nature of the chain. The chain consists of links which can pivot at the roller centers. Most of the chain weight is concentrated at the roller centers.

Further, if the position of each roller center is defined, the chain is completely located; there is a finite number of positions, a finite number of roller centers, and thus there is a finite number of degrees of freedom. The case will be taken in which the vibration displacements are so small that the pull along the chain is constant and equal to T, the strand vibrates only in one plane (in a plane parallel to the plane of the sprockets), and there is no damping of the vibrations.

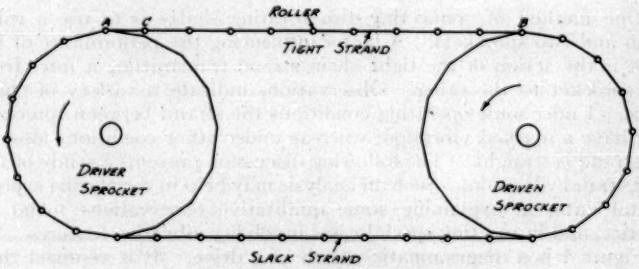

FIG 1. Diagrammatic sketch of a roller-chain drive.

The following notation will be used:

f = frequency of vibration, cycles per unit time,
g = gravitational acceleration,
M = mass of each load, equals weight/g,
N = total number of masses on loaded string,
P = chain pitch,
R = number of particular mass on string, R ranges from 1 to N,
T = tension in the tight chain strand,
t = time, and
y_R = vertical displacement of Rth particle from equilibrium position.

NATURAL FREQUENCIES OF VIBRATION.

Figure 2(a) shows a loaded string in the equilibrium position. The tension is T, and there are N different loads, each of mass M (M equals weight/g). Figure 2(b) shows three of the masses displaced from the equilibrium position. The distance y_R is the displacement of the Rth particle; R can have any integer value from 1 to N. As illustrated in

Fig. 2(*c*) the *R*th particle is acted upon by two forces, each of magnitude T, but at different directions. The force causing the acceleration of the mass is the sum of the vertical components of the two T forces.

Since force equals mass times acceleration, the basic equation for the *R*th particle is

$$M\frac{d^2y_R}{dt^2} = -\left[\frac{T(y_R - y_{R-1})}{\sqrt{(y_R - y_{R-1})^2 + P^2}} + \frac{T(y_R - y_{R+1})}{\sqrt{(y_R - y_{R+1})^2 + P^2}}\right]. \quad (1)$$

The case is taken in which each of the vertical displacement differences

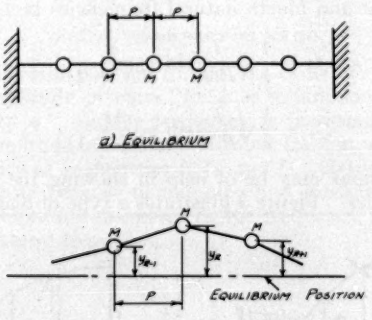

FIG. 2. Positions of loaded string.

$(y_R - y_{R-1})$ and $(y_R - y_{R+1})$ is small in comparison with P. Then Eq. 1 becomes

$$\frac{d^2y_R}{dt^2} = \frac{T}{MP}[y_{R-1} - 2y_R + y_{R+1}]. \quad (2)$$

Equation 2 is the equation of motion for the *R*th particle. There are N such equations, for $R = 1$, $R = 2$, $\cdots$ $R = N$. These equations must be solved simultaneously. The solution to types of these N differential equations of motion can be found in reference books on vibrations and mathematical physics.[3] We are here concerned with the engineering

[3] See, for example, "Theory of Vibrating Systems and Sound," by I. B. Crandall, D. Van Nostrand, New York, 1927, p. 64; and "Principles of Mathematical Physics," by William V. Houston, McGraw-Hill Book Co., Inc., New York, p. 124.

applications of these solutions. One final result is the relation for each natural frequency of vibration f_s,

$$f_s = \frac{1}{\pi}\sqrt{\frac{T}{PM}} \sin \frac{s\pi}{2(N+1)} \tag{3}$$

where s is any integer from 1 to N.

For a certain chain strand with N roller centers between the supporting ends there are N natural frequencies of vibration. The number N does not include the end rollers bedded on the sprockets. As an example, the first and fourth natural frequencies become, respectively

$$f_1 = \frac{1}{\pi}\sqrt{\frac{T}{PM}} \sin \frac{\pi}{2(N+1)}, \tag{4}$$

$$f_4 = \frac{1}{\pi}\sqrt{\frac{T}{PM}} \sin \frac{4\pi}{2(N+1)}. \tag{5}$$

Certain diagrams may be of help in showing the variation of the natural frequencies. Figure 3 illustrates a type of diagram which may

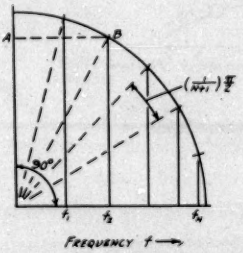

FIG. 3. Frequency spectrum.

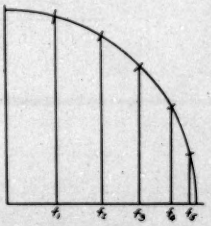

FIG. 4. Frequency spectrum for case with five masses.

be called a "frequency spectrum." The right angle is divided into divisions equal to the angle

$$\left(\frac{1}{1+N}\right)\frac{\pi}{2}.$$

Then each horizontal projected distance is proportional to a certain frequency. For example, the line AB is proportional to

$$\sin\left(\frac{2}{N+1}\right)\frac{\pi}{2},$$

and thus to f_2. Figure 4 illustrates a spectrum for five masses.

Usually there is a relatively large number of rollers in the chain strand. A specific example with a small number, however, will help

to bring out some basic ideas. Figure 5 illustrates a case with three masses. Consider a vibrating force $F \sin 2\pi ft$ acting on mass No. 1. The factor F is a constant, t is time, $2\pi f$ is an angular speed in radians per second, and $2\pi ft$ is an angle proportional to time. At each frequency there is a certain displacement amplitude (maximum displacement from equilibrium position during one cycle of operation). The displacement amplitude of mass No. 1 as a function of frequency is illustrated in Fig. 6.[4] As the frequency is increased from zero, the displacement amplitude increases. Without friction, the displacement approaches infinity at frequencies f_1, f_2, and f_3. Actually the displacement is not infinite because of damping and mechanical constraints. Note that the amplitude decreases as the frequency increases beyond f_3, the highest natural frequency.

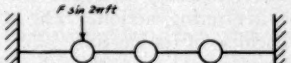

FIG. 5. System with three masses.

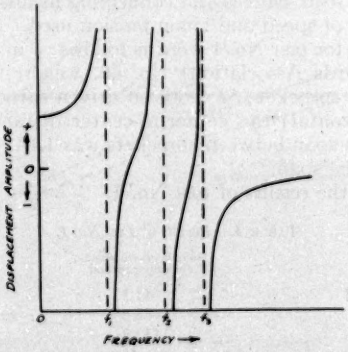

FIG. 6. Displacement amplitude of mass No. 1 in a system of three masses.

Referring to Fig. 1, after roller A seats on the sprocket, the link connecting rollers A and C starts to turn or articulate. After an interval, roller C will meet the sprocket and there will be an impact. The action during impact is complicated and has not been completely established yet. An impact or impressed frequency close to one of the natural frequencies may cause a marked vibration of the strand. Equa-

[4] The details for this particular case are given in "Mechanical Vibrations," by J. P. Den Hartog, McGraw-Hill Book Co., Inc., New York, 1940, p. 151.

tion 3 shows that the highest natural frequency is given when $s = N$, or

$$f_N = \frac{1}{\pi}\sqrt{\frac{T}{PM}}\sin\frac{N\pi}{2(N+1)}. \qquad (6)$$

For a large number of rollers the sine term in Eq. 4 may be close to unity. If an impressed frequency is above the highest natural frequency, there will be a filtering action. The strand will not follow the rapid movement because of its inertia.

Thus the chain strand may have different actions depending upon the impressed frequency and the natural frequency. It would be desirable to have further information as to damping and the effect of some distribution of link weights. The foregoing simplified analysis, however, gives simple relations which can provide some quantitative information. These relations might be modified by coefficients as more data become available.

LABORATORY TESTS.

Two laboratory tests were made for the purpose of comparing experimental and theoretical results. Special precautions were taken to eliminate torsional or other cyclic actions which might have occurred in the driving and load sources and connecting means with the improbable combinations of speed and chain tension used.

The conditions for test No. 1 were as follows: ½-in. pitch chain, ASA (American Standards Association) No. 40, weight 0.41 lb. per foot, 14 teeth on driver sprocket; 52 teeth on driven sprocket; the sprocket centers were horizontal; the center-to-center distance was 1.205 ft.; the length of chain span between sprockets was 1.18 ft.; and there were 28 free rollers in the span.

Table I shows the results of test No. 1.

TABLE I.—*Results of Test No. 1.*

Mode	Calculated frequency, cycles per second	Observed frequency, cycles per second
Fundamental	44.3	44.3
Second	88.6	90.1
Third	132.9	135.0

Table I shows fair agreement between the calculated frequency of vibration and the observed frequency. In all three cases the observed frequency of vibration equalled the tooth-impact frequency or product of sprocket angular speed times the number of teeth. In the fundamental mode there was a displacement node at each end of the strand and a maximum displacement at the center. In the second mode there was a displacement node at the two ends and the middle. In the third mode there was a displacement node at each end of the strand and two displacement nodes in between the ends of the strand.

The conditions for test No. 2 were as follows: 12 teeth on each sprocket; the sprocket centers were horizontal; the center-to-center distance was 1.167 ft.; the length of chain span between sprockets was 1.167 ft.; there were 13 free rollers in the span; and the chain was double pitch, 1-in., ASA No. 2040, except it had $\frac{5}{8}$-in. diameter rollers. This is a 1-in. pitch chain with sideplate cross-sectional area roughly equivalent to the ASA No. 40 chain and using the same pins and bushings as the ASA No. 40 chain. This particular chain was assembled with rollers with $\frac{5}{8}$-in. outside diameter instead of the conventional $\frac{5}{16}$-in. outside diameter. Weight per foot of chain was 0.502 lb.

The chain tension and sprocket angular speed were changed to give various combinations which would produce a fundamental mode of vibration in the driving span of chain.

The experimental points in Fig. 7 show combinations of chain ten-

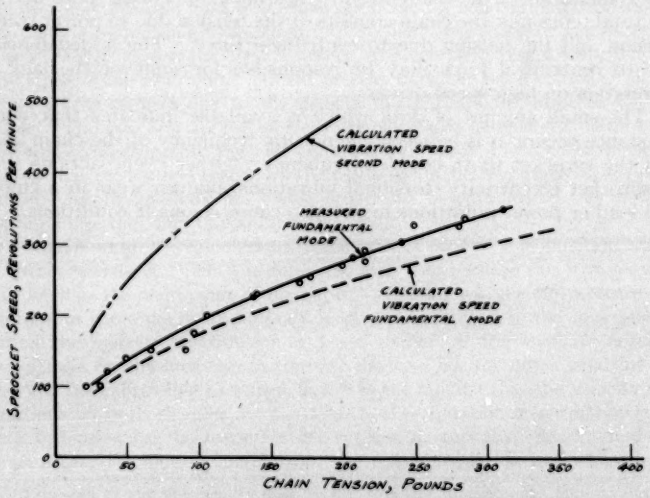

FIG. 7. Plot of sprocket speed *versus* chain tension for chain vibration.

sion and angular speed which produced a fundamental mode of vibration. Also shown (broken line) is the calculated fundamental and second modes of vibration for comparison. Experimental results show that for a given tension, the observed fundamental mode occurred at a sprocket speed 14 per cent in excess of the calculated speed. As in test No. 1; the frequency of vibration equalled the tooth-impact frequency.

The weight distribution in the two chains used in the two tests differs in that considerably greater mass is located around the chain pin in test No. 2 than in test No. 1.

In addition to the foregoing tests, further experiments were made on $\frac{3}{8}$-in. pitch chain, ASA No. 50, in an effort to correlate chain-strand vibrations with some cyclic phenomena on a 50-hp. chain-test dynamometer.

In one test it was found that the frequency of the fundamental vibration mode of the chain strand was 14 per cent in excess of the calculated frequency. As indicated in the previous tests, the forcing frequency appears to be that of tooth contact. Several other tests have failed to show any correlation between chain-strand vibration and tooth-impact frequency, sprocket speed, chain speed, or other known cyclic actions. There are innumerable factors present in this test arrangement which might affect the chain-strand vibrations. For high speed drives, the total tension in the chain consists of the tension due to power transmission and the tension due to centrifugal force. The added tension due to centrifugal force may be responsible for some of the lack of correlation on high speed drives.

The small amount of data which is available indicates that when resonance occurs, it is between the natural frequency of the chain span and the sprocket tooth contact frequency. Other cyclic actions, such as sprocket eccentricity, torsional vibrations, uneven wear in a chain, and load or power pulsations might also cause resonant conditions.

ON THE TWO-DIMENSIONAL STEADY TURBULENT FLOW OF A COMPRESSIBLE FLUID FAR BEHIND A SOLID SYMMETRICAL BODY. I. (MOMENTUM TRANSFER THEORY).[1]

BY

M. Z. KRZYWOBLOCKI, Dr. Ae. E., M.A., Dipl. Ing.[2]

INTRODUCTION.

Schlichting [3] solved the equations of a steady turbulent flow of an incompressible fluid in a wake far behind a solid symmetrical body by application of the method of successive approximations. In the present paper this method was extended to apply to the solution of the equations of a steady turbulent flow of a compressible fluid in a wake far behind a solid symmetrical body. The following equations were taken into account: equations of motion, continuity, energy, and state. As is known, the complete tensor form for the eddy stress is due to Prandtl. In the present work two other complete tensor forms for the stresses due to the density fluctuations are proposed by the author. The two first approximations are calculated completely. From the third approximation only the first equation was taken into account in order to show that the method used works perfectly to any desired degree of approximation. Approximations of a higher order may be obtained by this method and are a question only of routine work.

Throughout all the calculations in the present paper the momentum transfer theory was used. This was done purposely and perhaps this item may be considered as the main point of the paper. As is known, the temperature distribution in the wake behind a body is closer to reality if in the equations of motion the velocity distribution is based rather on the vorticity transfer theory than on the momentum transfer theory. On the other hand, in the theory of incompressible turbulent flow it is the opinion that the velocity distributions by both theories are the same.[4] The question arises whether the same statement can be made for a compressible turbulent flow. Here the temperature distribution found from the energy equation undoubtedly in-

[1] Presented at the Summer Meeting of the American Mathematical Society, Applied Mathematics Section, Yale University, September 2–5, 1947.

[2] Associate Professor, Department of Aeronautical Engineering, University of Illinois, Urbana, Ill.

[3] H. Schlichting, "Über das ebene Windschatten Problem," *Ingenieur–Archiv*, Vol. I., 1930, pp. 533–571.

[4] See: "Modern Developments in Fluid Dynamics," Edited by S. Goldstein, Oxford, Clarendon Press, 1938, Vol. II, p. 585. Let us mention that this remark refers only to a two-dimensional flow pattern. In a three-dimensional flow problem both theories may give incorrect results. See: S. Tomotika, "On the Velocity and Temperature Distributions in the Turbulent Wake Behind a Heated Body of Revolution," *Proc. Royal Soc. London*, Series A, Vol. 165, 1938, p. 53.

fluences the velocity distribution through the equation of state and varying density. To answer this question, in any case from a theoretical point of view, the author sees the following method: to solve the entire system of equations based on both theories. The present paper solves the first part.[5] The comparison and inspection of both solutions will show the largeness of existing discrepancies.

It is a well known fact that the phenomenon of turbulence refers to a three-dimensional space and, strictly speaking, should not be treated in a two-dimensional plane. However, if we treat the problem from a general point of view in two dimensions and do not go into details, it is possible to assume that the body is infinitely long, that the disturbances caused by the body extend to infinity in two perpendicular directions of a horizontal plane. This assumption permits one up to a certain extent to treat the problem in a two-dimensional plane, particularly when one intends to obtain only certain general conclusions. Of course, the following question may arise: is a two-dimensional, artificial turbulent flow pattern identical with a two-dimensional cross-section of an actual three-dimensional turbulent flow pattern, even if the body extends to infinity? The answer may be obtained by several approaches such as experimental, etc. But the two flows will not be the same. Moreover, even a series of two-dimensional cross-sections, performed simultaneously at various points along the axis of the cylindrical body, will show different flow patterns. These flow patterns will, however, be statistically identical. Nevertheless, following the works of many previous authors on this subject, intending to draw only the most general conclusions and keeping in mind that the simplicity factor should also not be neglected, in the present paper we shall treat the problem as a two-dimensional one. In other words, the two-dimensional approach will be considered as the first approximation to the actual, three-dimensional flow pattern. Similarly, although the turbulent motion is an unsteady motion, assuming that the variation of all the dependent variables with respect to the time can be neglected, we shall treat in the analytical approach the motion as a steady one.

I. THE GENERAL EQUATIONS OF MOTION AND BASIC LAWS.

The general equations of motion of a compressible viscous fluid in a three-dimensional system of coordinates (with the equation of continuity included) are:

$$\begin{aligned}
&\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho uv) + \partial_z(\rho uw) = \rho X + \partial_x(p_{xx}) + \partial_y(p_{yx}) + \partial_z(p_{zx}),\\
&\partial_t(\rho v) + \partial_x(\rho vu) + \partial_y(\rho v^2) + \partial_z(\rho vw) = \rho Y + \partial_x(p_{xy}) + \partial_y(p_{yy}) + \partial_z(p_{zy}),\\
&\partial_t(\rho w) + \partial_x(\rho wu) + \partial_y(\rho wv) + \partial_z(\rho w^2) = \rho Z + \partial_x(p_{xz}) + \partial_y(p_{yz}) + \partial_z(p_{zz}),
\end{aligned} \qquad (1)$$

[5] The second part will consist of a separate paper to be published in the February issue of this JOURNAL. It will contain the solution of the entire system of equations based on the vorticity transfer theory.

where the symbols used denote:

$$\begin{aligned} p_{xx} &= -p - \tfrac{2}{3}\mu(\partial_x u + \partial_y v + \partial_z w) + 2\mu\partial_x u, \\ p_{yy} &= -p - \tfrac{2}{3}\mu(\partial_x u + \partial_y v + \partial_z w) + 2\mu\partial_y v, \\ p_{zz} &= -p - \tfrac{2}{3}\mu(\partial_x u + \partial_y v + \partial_z w) + 2\mu\partial_z w, \end{aligned} \tag{1a}$$

$$\begin{aligned} p_{yz} &= p_{zy} = \mu(\partial_y w + \partial_z v), \\ p_{zx} &= p_{xz} = \mu(\partial_z u + \partial_x w), \\ p_{xy} &= p_{yx} = \mu(\partial_x v + \partial_y u). \end{aligned}$$

The subscripts below the differentiation symbols denote the independent variables relative to which the differentiation is taken.[6] Besides (1) the equation of continuity must be taken into account:

$$\partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0. \tag{2}$$

Equation of conservation of energy:[7]

$$\rho \mathcal{J} c_v DT + p(\partial_x u + \partial_y v + \partial_z w) = \mathcal{J}\nabla^2 T + \phi, \tag{3}$$

$$DT = \partial_t T + u\partial_x T + v\partial_y T + w\partial_z T, \tag{3a}$$

$$\nabla^2 T = \partial_x(k\partial_x T) + \partial_y(k\partial_y T) + \partial_z(k\partial_z T), \tag{3b}$$

$$\begin{aligned} \phi = \mu\{2[(\partial_x u)^2 + (\partial_y v)^2 + (\partial_z w)^2] - \tfrac{2}{3}(\partial_x u + \partial_y v + \partial_z w)^2 \\ + (\partial_y w + \partial_z v)^2 + (\partial_z u + \partial_x w)^2 + (\partial_x v + \partial_y u)^2\}. \end{aligned} \tag{3c}$$

The symbols denote:

t—time,
T—absolute temperature,
$\mathcal{J}$—mechanical equivalent of heat,
c_v—specific heat at constant volume,
ϕ—dissipation function,
k—coefficient of thermal conductivity,
$\vec{F}$—external force per unit mass,
p—pressure,
X, Y, Z are the components of the external force per unit mass, which includes all the impressed forces. We put:

$$F = -\operatorname{grad}\Omega + \vec{G}, \qquad \vec{G} = \vec{g}\,\frac{(\rho - \rho_0)}{\rho}. \tag{4}$$

The force $\vec{G}$ arises from the effect of gravity on the fluid of variable density. Making use of the equation of continuity multiplied by $\mathcal{J}Tc_v$ permits Eq. 3 to

[6] The signs in Eq. 1a are in accordance with those given in: "Modern Developments in Fluid Dynamics," edited by S. Goldstein, Oxford, Clarendon Press, 1938. In the work: W. F. Durand, "Aerodynamic Theory," Berlin, Julius Springer, 1936, Vol. VI, H. L. Dryden, "Aerodynamics of Cooling," p. 239, the signs used in expressions for p_{ij} are opposite to those used in Goldstein's work.

[7] Goldstein, *loc. cit.*, Vol. II, p. 606.

be written in the form:

$$T c_v[\partial_t(\rho T) + \partial_x(u\rho T) + \partial_y(v\rho T) + \partial_z(w\rho T)] + p(\partial_x u + \partial_y v + \partial_z w) = T\nabla^2 T + \phi. \quad (5)$$

The equation of state for ideal gases is:

$$p = R\rho T. \quad (6)$$

Let us apply the averaging process to all the equations cited above.[8] The following assumptions are made: the turbulent motion can be separated into a fictitious mean flow whose components are $\bar{u}$, $\bar{v}$, $\bar{w}$. The mean motion of a fictitious fluid with the average density $\bar{\rho}$ obeys the law of conservation of mass. On this mean motion there is superposed a turbulent motion. This relative motion has the component velocities u', v', w', the mean values of which are zero. Subject to fluctuations are velocities, density, temperature and the pressure.

In the most general case of flow of a compressible viscous fluid the coefficients μ, k, c_p and c_v are functions of pressure and temperature. Assuming, however, the invariability of the last two, one cannot neglect the variability of the first two coefficients. Since the coefficient k may be considered to be proportional to a product of μ and c_v, it is sufficient to estimate the variation of μ only. Because of the fact that the dependence of this coefficient on the pressure is very small for any change likely to occur in ordinary circumstances [9] one may take into account only the dependence on the temperature. Hence, the coefficients μ and k will also be subject to fluctuations together with the temperature.

The results of the averaging are the following:

$$\begin{aligned}
\bar{\rho} D\bar{u} &= \bar{\rho}X - \partial_t M_x - \partial_x \pi_{xx} - \partial_y \pi_{yx} - \partial_z \pi_{zx}, \\
\bar{\rho} D\bar{v} &= \bar{\rho}Y - \partial_t M_y - \partial_x \pi_{xy} - \partial_y \pi_{yy} - \partial_z \pi_{zy}, \\
\bar{\rho} D\bar{w} &= \bar{\rho}Z - \partial_t M_z - \partial_x \pi_{xz} - \partial_y \pi_{yz} - \partial_z \pi_{zz},
\end{aligned} \quad (7)$$

$$\begin{aligned}
D\bar{u} &= \partial_t \bar{u} + \bar{u}\partial_x \bar{u} + \bar{v}\partial_y \bar{u} + \bar{w}\partial_z \bar{u}, \\
D\bar{v} &= \partial_t \bar{v} + \bar{u}\partial_x \bar{v} + \bar{v}\partial_y \bar{v} + \bar{w}\partial_z \bar{v}, \qquad \text{etc.}, \\
M_x &= \overline{\rho' u'}, \qquad M_y = \overline{\rho' v'}, \qquad M_z = \overline{\rho' w'}.*
\end{aligned}$$

[8] The laws of the averaging process are the following:

$$\varphi = \bar{\varphi} + \varphi', \qquad \bar{\bar{\varphi}} = \bar{\varphi}, \qquad \overline{\bar{\varphi}\bar{\psi}} = \bar{\varphi}\bar{\psi}, \qquad \overline{\varphi'} = 0, \qquad \overline{\varphi\psi} = \bar{\varphi}\bar{\psi} + \overline{\varphi'\psi'},$$
$$\overline{\varphi\psi\chi} = \bar{\varphi}\bar{\psi}\bar{\chi} + \bar{\varphi}\overline{\psi'\chi'} + \bar{\psi}\overline{\varphi'\chi'} + \bar{\chi}\overline{\varphi'\psi'} + \overline{\varphi'\psi'\chi'},$$
$$\overline{\partial_x \varphi} = \partial_x \bar{\varphi}, \qquad \overline{\partial_t \varphi} = \partial_t \bar{\varphi}, \text{ etc.}$$

For more details see Goldstein, *loc. cit.*, Vol. I, p. 191.

[9] Goldstein, *loc. cit.*, Vol. I., p. 7.

* The values of all the other coefficients and terms are given in the Appendix. The set of equations (7) differs considerably from the one cited in: H. L. Dryden, F. D. Murnaghan, H. Bateman, "Hydrodynamics," *Bull. Nat. Research Council*, February, 1932, No. 84, Washington, p. 573, Eq. (4.12′). The author could not find out the reason for these discrepancies. But a statement must be made that several equations in the "Compressible Fluids" part of this Bulletin are incorrect. See the remark below.

The equation of continuity of the fictitious mean motion with the average density $\bar{\rho}$ is:

$$\partial_t\bar{\rho} + \partial_x(\bar{\rho}\bar{u}) + \partial_y(\bar{\rho}\bar{v}) + \partial_z(\bar{\rho}\bar{w}) = 0. \tag{8}$$

The averaged equation of energy becomes:

$$\mathcal{J}c_v(\bar{K} + K' + L) + \bar{N} = \mathcal{J}(\bar{P} + P') + \bar{R} + R' + S.^{**} \tag{9}$$

Equation of state:

$$\bar{p} = R(\bar{\rho}\bar{T}), \qquad \bar{p}' = R\overline{\rho'T'} = 0. \tag{10}$$

II. ADDITIONAL STRESSES.

As is known and can be easily seen, the equations of motion of the mean flow are the same as ordinary equations of motion provided that certain stress components due to the fluctuations are added to the mean values of the stresses which are due to viscous forces. These virtual additional stresses may be considered to be composed of two parts: Reynolds' stresses (eddy stresses) and stresses due to the density fluctuations. Reynolds' stresses are the stresses of the second order [10] and they are the mathematical representations of the transport of momentum across a surface due to the velocity fluctuations. Similarly the stresses due to the density fluctuations are the stresses of the second and third order and they may be considered to be the mathematical representations of the transport of momentum across a surface due to the density fluctuations.[11] According to the mixture length theory, the effect of turbulence on the transfer of a property is represented by:

$$Q = -l'V\frac{d\theta}{dy},^{***} \tag{11}$$

where the symbols denote:

Q = mean rate of transfer of θ across a unit area perpendicular to y,
l' = mixture length,
θ = the property of the gas,
U, V, W = the root-mean-square values of the fluctuations u', v', w', relatively that is:

$$U = \sqrt{\overline{u'^2}}, \qquad V = \sqrt{\overline{v'^2}}, \qquad W = \sqrt{\overline{w'^2}}. \tag{12}$$

As is known, to account for the distribution of mean velocity in turbulent incompressible fields of flow, the hypothesis that momentum is a transferable

** The author purposely put down these equations in full extension in the Appendix. In the work by George Porter Paine, "Aerodynamics of the Psychrometer," *Delaware Notes*, University of Delaware, Newark, Delaware, p. 40, the equation of energy (14) is completely incorrect. One may easily verify that in Eq. 14 the dimensions of terms are different. Similarly Eq. 19 on p. 42 which constituted the basis of the ensuing study has terms of different dimensions. Equation 14 is cited in: Dryden, Murnaghan, Bateman "Hydrodynamics," *loc. cit.*, pp. 576, 577.

[10] The order is counted by number of components (u', v', ρ', etc.) due to the fluctuations entering each term.

[11] For example: $\bar{u}\,\overline{\rho'u'}$ is the momentum $\rho'\bar{u}$ transferred by u'.

*** Goldstein, *loc. cit.*, p. 206.

property in the sense of the equation mentioned above had been forward. Also as is known, in this theory the effect of fluctuating pressure gradients is neglected. That enables us to write $\bar{p}\bar{u}$ instead of θ. Q then represents the rate of transfer of momentum in the y-direction. Hence the general diatensor (unsymmetrical) for eddy stress in this case becomes:

$$(\tau) = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} = -\bar{\rho} \begin{bmatrix} \overline{u'u'} & \overline{v'u'} & \overline{w'u'} \\ \overline{u'v'} & \overline{v'v'} & \overline{w'v'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'} \end{bmatrix}, \tag{13}$$

or

$$(\tau) = \bar{\rho}l' \begin{bmatrix} \partial_x\bar{u}U & \partial_x\bar{v}U & \partial_x\bar{w}U \\ \partial_y\bar{u}V & \partial_y\bar{v}V & \partial_y\bar{w}V \\ \partial_z\bar{u}W & \partial_z\bar{v}W & \partial_z\bar{w}W \end{bmatrix}. \tag{13a}$$

$\bar{\rho}\,\overline{u'u'}$, $\bar{\rho}\,\overline{v'v'}$, $\bar{\rho}\,\overline{w'w'}$, are eddy normal stresses, and $\bar{\rho}\,\overline{v'u'}$, $\bar{\rho}\,\overline{w'u'}$, $\bar{\rho}\,\overline{u'v'}$, etc. are eddy shearing stresses. In an isotropic turbulence $U = V = W$. For an absolute correlation [13] and an isotropic turbulence we have: $\partial_x\bar{v}U = \partial_y\bar{u}V$, $\partial_x\bar{w}U = \partial_z\bar{u}W$, etc. and the diatensor (13) and (13a) changes into a (symmetrical) tensor.

The items explained above are known. They were presented in order to show the application of this idea to diatensors for stresses due to the density fluctuations. The author proposes the following diatensor for the stresses of the second order due to the density fluctuations:

$$(\sigma) = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = -\begin{bmatrix} 2\bar{u}\,\overline{u'\rho'} & \bar{u}\,\overline{v'\rho'}+\bar{v}\,\overline{u'\rho'} & \bar{u}\,\overline{w'\rho'}+\bar{w}\,\overline{u'\rho'} \\ \bar{v}\,\overline{u'\rho'}+\bar{u}\,\overline{v'\rho'} & 2\bar{v}\,\overline{v'\rho'} & \bar{v}\,\overline{w'\rho'}+\bar{w}\,\overline{v'\rho'} \\ \bar{w}\,\overline{u'\rho'}+\bar{u}\,\overline{w'\rho'} & \bar{w}\,\overline{v'\rho'}+\bar{v}\,\overline{w'\rho'} & 2\bar{w}\,\overline{w'\rho'} \end{bmatrix}, \tag{14}$$

or

$$(\sigma) = l' \begin{bmatrix} 2\bar{u}U\partial_x\bar{\rho} & (\bar{u}V+\bar{v}U)\partial_x\bar{\rho} & (\bar{u}W+\bar{w}U)\partial_x\bar{\rho} \\ (\bar{v}U+\bar{u}V)\partial_y\bar{\rho} & 2\bar{v}V\partial_y\bar{\rho} & (\bar{v}W+\bar{w}V)\partial_y\bar{\rho} \\ (\bar{w}U+\bar{u}W)\partial_z\bar{\rho} & (\bar{w}V+\bar{v}W)\partial_z\bar{\rho} & 2\bar{w}W\partial_z\bar{\rho} \end{bmatrix}. \tag{14a}$$

Putting forward the hypothesis that in an isotropic turbulence the variations of density in all three directions are equal, changes the diatensor (14) into a symmetrical tensor. In a similar way the proposal is made for the diatensor

[13] The coefficient of correlation between u' and v' is defined as:

$$R_{u'v'} = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}} = \frac{\overline{u'v'}}{UV}.$$

For an absolute correlation $R_{u'v'} = 1$. For an absolute correlation and an isotropic turbulence we have: $|\overline{u'v'}| = UV = U^2 = V^2$ or $u' = v'$. In further considerations also: $u' = v' = w'$. Hence also: $\partial_x\bar{u} = \partial_y\bar{u} = \partial_z\bar{u} = \partial_x\bar{v} = \partial_y\bar{v} = \cdots$.

for the stresses of the third order due to the density fluctuations to be:

$$(\Pi) = \begin{Vmatrix} \Pi_{xx} & \Pi_{yx} & \Pi_{zx} \\ \Pi_{xy} & \Pi_{yy} & \Pi_{zy} \\ \Pi_{xz} & \Pi_{yz} & \Pi_{zz} \end{Vmatrix} = - \begin{Vmatrix} \overline{u'u'\rho'} & \overline{v'u'\rho'} & \overline{w'u'\rho'} \\ \overline{u'v'\rho'} & \overline{v'v'\rho'} & \overline{w'v'\rho'} \\ \overline{u'w'\rho'} & \overline{v'w'\rho'} & \overline{w'w'\rho'} \end{Vmatrix}, \tag{15}$$

or

$$(\Pi) = r' \begin{Vmatrix} UU\partial_x\bar{\rho} & VU\partial_x\bar{\rho} & WU\partial_x\bar{\rho} \\ UV\partial_y\bar{\rho} & VV\partial_y\bar{\rho} & WV\partial_y\bar{\rho} \\ UW\partial_z\bar{\rho} & VW\partial_z\bar{\rho} & WW\partial_z\bar{\rho} \end{Vmatrix}. \tag{15a}$$

For an isotropic turbulence we obtain a symmetrical tensor.

III. COMPLETE TENSOR FORMS.

A complete tensor form for the eddy stress is due to Prandtl.[13] It is:

$$(\tau) = \bar{\rho}l'^2\overline{\mathcal{T}}(\nabla\vec{v} + \vec{v}\nabla), \tag{16}$$

where the symbols denote:

$\overline{\mathcal{T}}^2$ = quadratic invariant of the deformation tensor multiplied by (-1),
$\nabla\vec{v}$ = dyadic (affinor) of v,
$\vec{v}\nabla$ = conjugate dyadic of v.

Putting:

$$\begin{aligned} \nabla\vec{v} &= \vec{i};\vec{A} + \vec{j};\vec{B} + \vec{k};\vec{C}, \\ \vec{A} &= \vec{i}\partial_x\bar{u} + \vec{j}\partial_y\bar{u} + \vec{k}\partial_z\bar{u}, \\ \vec{B} &= \vec{i}\partial_x\bar{v} + \vec{j}\partial_y\bar{v} + \vec{k}\partial_z\bar{v}, \\ \vec{C} &= \vec{i}\partial_x\bar{w} + \vec{j}\partial_y\bar{w} + \vec{k}\partial_z\bar{w}, \\ \vec{v}\nabla &= \vec{A};\vec{i} + \vec{B};\vec{j} + \vec{C};\vec{k}, \end{aligned} \tag{16a}$$

we obtain in the known way:

$$\begin{aligned} \nabla\vec{v} = {} & \partial_x\bar{u}\vec{i};\vec{i} + \partial_y\bar{u}\vec{i};\vec{j} + \partial_z\bar{u}\vec{i};\vec{k} + \\ & + \partial_x\bar{v}\vec{j};\vec{i} + \partial_y\bar{v}\vec{j};\vec{j} + \partial_z\bar{v}\vec{j};\vec{k} + \\ & + \partial_x\bar{w}\vec{k};\vec{i} + \partial_y\bar{w}\vec{k};\vec{j} + \partial_z\bar{w}\vec{k};\vec{k}. \end{aligned} \tag{16b}$$

Expanding the same manner $\vec{v}\nabla$ we obtain the final form of the complete tensor:

$$(\tau) = \bar{\rho}l'^2\overline{\mathcal{T}}\begin{pmatrix} 2\partial_x\bar{u} & \partial_y\bar{u} + \partial_x\bar{v} & \partial_z\bar{u} + \partial_x\bar{w} \\ \partial_x\bar{v} + \partial_y\bar{u} & 2\partial_y\bar{v} & \partial_z\bar{v} + \partial_y\bar{w} \\ \partial_z\bar{w} + \partial_x\bar{u} & \partial_y\bar{w} + \partial_z\bar{v} & 2\partial_z\bar{w} \end{pmatrix}. \tag{16c}$$

The quadratic invariant of this deformation tensor multiplied by (-1) gives

[13] See: H. Schlichting's paper, *loc. cit.*, p. 533, Goldstein's work, *loc. cit.*, p. 208. Also: Walter Tollmien, "Berechnung Turbulenter Ausbreitungsvorgänge," *ZAMM.*, Vol. 6, No. 6, Dec., 1926, pp. 468–478; Walter Tollmien, "Turbulente Strömungen," *Handbuch der Experimentalphysik*, Vol. 4, Part 1, Leipzig, 1931, p. 289; L. M. Swain, "On the Turbulent Wake Behind a Body of Revolution," *Proc. Royal Soc. London*, Series A, Vol. 125, 1929, p. 647; Dryden, Murnaghan, Bateman, "Hydrodynamics," *loc. cit.*, p. 397.

the value of $\overline{T}^2$:

$$\overline{T}^2 = (\partial_x\bar{v} + \partial_y\bar{w})^2 + (\partial_x\bar{u} + \partial_z\bar{w})^2 + (\partial_y\bar{u} + \partial_z\bar{v})^2 - 4(\partial_y\bar{v}\partial_z\bar{w} + \partial_z\bar{w}\partial_x\bar{u} + \partial_x\bar{u}\partial_y\bar{v}). \quad (17)$$

Adding to (17) the square of the continuity equation in a steady flow with $\bar{\rho}$ = constant, that is, the expression $2(\partial_x\bar{u} + \partial_y\bar{v} + \partial_z\bar{w})^2$, we obtain another form of $\overline{T}^2$, used sometimes:

$$\overline{T}^2 = 2[(\partial_x\bar{u})^2 + (\partial_y\bar{v})^2 + (\partial_z\bar{w})^2] + (\partial_z\bar{v} + \partial_y\bar{w})^2 + (\partial_z\bar{u} + \partial_x\bar{w})^2 + (\partial_y\bar{u} + \partial_x\bar{v})^2. \quad (17a)$$

The author proposes the complete tensor forms for the second and third order stresses due to the density fluctuations. They may be written in the following symbolical way:

Second order:

$$(\sigma) = l'^2[\nabla(v^2 \operatorname{grad} \bar{\rho}) + (v^2 \operatorname{grad} \bar{\rho})\nabla], \quad (18)$$

$$\nabla(v^2 \operatorname{grad} \bar{\rho}) = \vec{i}\partial_x\bar{\rho}; \vec{D} + \vec{j}\partial_y\bar{\rho}; \vec{E} + \vec{k}\partial_z\bar{\rho}; \vec{G}, \quad (18a)$$

$$\vec{D} = \vec{i}\partial_x(\bar{u}^2) + \vec{j}\partial_x(\bar{u}\bar{v}) + \vec{k}\partial_x(\bar{u}\bar{w}),$$
$$\vec{E} = \vec{i}\partial_y(\bar{v}\bar{u}) + \vec{j}\partial_y(\bar{v}^2) + \vec{k}\partial_y(\bar{v}\bar{w}),$$
$$\vec{G} = \vec{i}\partial_z(\bar{w}\bar{u}) + \vec{j}\partial_z(\bar{w}\bar{v}) + \vec{k}\partial_z(\bar{w}^2).$$

Third order:

$$(\Pi) = r'^3[\nabla(dv)^2 \operatorname{grad} \bar{\rho} + (dv)^2 \operatorname{grad} \bar{\rho}\nabla], \quad (19)$$

$$\nabla(dv)^2 \operatorname{grad} \bar{\rho} = \vec{i}\partial_x\bar{\rho}; \vec{H} + \vec{j}\partial_y\bar{\rho}; \vec{K} + \vec{k}\partial_z\bar{\rho}; \vec{L},$$
$$\vec{H} = \vec{i}(\partial_x\bar{u})^2 + \vec{j}(\partial_x\bar{u})(\partial_x\bar{v}) + \vec{k}(\partial_x\bar{u})(\partial_x\bar{w}), \quad (19a)$$
$$\vec{K} = \vec{i}(\partial_y\bar{v})(\partial_y\bar{u}) + \vec{j}(\partial_y\bar{v})^2 + \vec{k}(\partial_y\bar{v})(\partial_y\bar{w}),$$
$$\vec{L} = \vec{i}(\partial_z\bar{w})(\partial_z\bar{u}) + \vec{j}(\partial_z\bar{w})(\partial_z\bar{v}) + \vec{k}(\partial_z\bar{w})^2.$$

Introducing the quadratic invariants $\overline{T}^2$ (multiplied by -1) of the deformations tensors we may write:

$$(\sigma) = l'^4[\bar{\rho}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)]^{-1}\overline{T}[\nabla(v^2 \operatorname{grad} \bar{\rho}) + (v^2 \operatorname{grad} \bar{\rho})\nabla], \quad (18b)$$

and

$$(\Pi) = r'^6[\bar{\rho}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)]^{-1}\overline{T}[\nabla(dv)^2 \operatorname{grad} \bar{\rho} + (dv)^2 \operatorname{grad} \bar{\rho}\nabla]. \quad (19b)$$

IV. TWO-DIMENSIONAL STEADY FLOW WITH NO EXTRANEOUS FORCES.[14]

When the boundary layer at the surface of a solid body becomes turbulent before it leaves the surface, the motion in the wake is turbulent. But even at lower Reynolds numbers, when a "free" laminar layer of vorticity exists behind the solid body, the motion at some distance downstream becomes turbulent. Also when a vortex-street is formed behind the obstacle, the motion sufficiently far downstream is completely turbulent. The velocity, density, temperature and pressure distribution at a considerable distance downstream in the wake behind a symmetrical body-cylinder (two-dimensional motion) may be calculated on either the momentum transfer or the vorticity transfer

[14] The z-axis is assumed to be vertical.

theory. In the present work the momentum transfer theory will be used. Following the assumptions of Reynolds' theory and of the momentum transfer theory [15] various simplifications will be introduced into our equations. Since the Reynolds' eddy stresses and stresses due to the density fluctuations are much greater than the stresses due to viscosity, except where the rate of deformation is very large, in all the equations of motion we shall omit all the terms expressing stresses due to viscosity. Following the assumptions of previous writers,[16] the heating due to pressure changes and to the dissipation of energy by viscosity will be neglected. Since in turbulent flow the effect of the eddy heat transfer terms is large compared with the effect of the molecular conductivity terms except where the temperature gradient is very large, we shall neglect all the molecular conductivity terms. (This is analogous to the largeness of the eddy stresses in comparison with the viscous stresses.)

Below, the calculations will be performed only for $y > 0$. The conditions for $y < 0$ may be obtained simply by reflexion with respect to x-axis. As the final result we obtain the following set of equations:

Two equations of motion:

$$\begin{aligned} \bar{\rho}\bar{u}\partial_x\bar{u} + \bar{\rho}\bar{v}\partial_y\bar{u} &= -\partial_x\pi_{xx} - \partial_y\pi_{yx}, \\ \bar{\rho}\bar{u}\partial_x\bar{v} + \bar{\rho}\bar{v}\partial_y\bar{v} &= -\partial_x\pi_{xy} - \partial_y\pi_{yy}, \end{aligned} \tag{20}$$

$$\begin{aligned} \pi_{xx} &= \bar{p} + \bar{\rho}\,\overline{u'u'} + 2\bar{u}\,\overline{\rho'u'} + \overline{\rho'u'^2}, \\ \pi_{yy} &= \bar{p} + \bar{\rho}\,\overline{v'v'} + 2\bar{v}\,\overline{\rho'v'} + \overline{\rho'v'^2}, \\ \pi_{yx} &= \pi_{xy} = \bar{\rho}\,\overline{u'v'} + \bar{u}\,\overline{\rho'v'} + \bar{v}\,\overline{\rho'u'} + \overline{\rho'u'v'}, \end{aligned} \tag{20a}$$

equation of continuity:

$$\partial_x(\bar{u}\bar{\rho}) + \partial_y(\bar{v}\bar{\rho}) = 0, \tag{21}$$

equation of energy:

$$\mathcal{J}c_v(\bar{K} + K' + L) + \bar{N} = 0,^{17} \tag{22}$$

$$\begin{aligned} \bar{K} &= \partial_x(\bar{u}\bar{\rho}\bar{T}) + \partial_y(\bar{v}\bar{\rho}\bar{T}) = \bar{u}\bar{\rho}\partial_x\bar{T} + \bar{v}\bar{\rho}\partial_y\bar{T}, \quad \text{with (21)}, \\ L &= \partial_x(\bar{u}\,\overline{\rho'T'} + \bar{\rho}\,\overline{u'T'} + \bar{T}\,\overline{u'\rho'}) + \partial_y(\bar{v}\,\overline{\rho'T'} + \bar{\rho}\,\overline{v'T'} + \bar{T}\,\overline{v'\rho'}), \\ K' &= \partial_x(\overline{u'\rho'T'}) + \partial_y(\overline{v'\rho'T'}), \\ \bar{N} &= \bar{p}(\partial_x\bar{u} + \partial_y\bar{v}) + \overline{p'\partial_x u'} + \overline{p'\partial_y v'}, \end{aligned} \tag{22a}$$

equation of state:

$$\bar{p} = R(\bar{\rho}\bar{T}). \tag{23}$$

The next step is to replace the values due to the fluctuations (u', v', ρ', T') by the values of fictitious mean flow ($\bar{u}$, $\bar{v}$, $\bar{\rho}$, $\bar{T}$). As is known, Prandtl assumed

[15] For more details see Goldstein's work, *loc. cit.*

[16] See Goldstein, *loc. cit.*, Chapter XV, Heat Transfer (Turbulent Flow), p. 646.

[17] In a compressible fluid the specific heat at constant pressure c_p should be used in the formulae instead of c_v. Hence below instead of c_v we shall use c_p. See Goldstein, *loc. cit.*, p. 646.

that:[18]

$$u' = l\partial_y\bar{u}, \qquad v' = l\partial_y\bar{u}, \qquad \tau = -\bar{\rho}\overline{u'v'} = \bar{\rho}l^2|\partial_y\bar{u}|\partial_y\bar{u}.^{19} \tag{24}$$

This assumption of Prandtl refers to the shearing stress. Below some assumptions have to be made concerning all the terms in the expressions for π_{xx}, π_{yy}, $\pi_{yz} = \pi_{zy}$. So let us assume that in all the expressions, the values u' and v' will be expressed by Prandtl's formula and ρ' by $l\partial_y\bar{\rho}$. It is reasonable to assume that the mixture length l is the same in all of the terms, since the same mechanism is at work. Hence we have:

$$\bar{u}\,\overline{\rho'v'} = -l^2\bar{u}|\partial_y\bar{u}|\partial_y\bar{\rho}, \qquad u'\rho' = v'\rho' = -l^2|\partial_y\bar{u}|\partial_y\bar{\rho}, \text{ etc.} \tag{25}$$

Similarly with Prandtl's hypothesis the temperature fluctuations may be expressed by the temperature of the fictitious mean flow:[20]

$$\bar{\rho}c_p\overline{v'T'} = -c_p\bar{\rho}\,\overline{Lv'}\partial_y\bar{T}, \tag{26}$$

where L is the length of the path of the particles. Since one may write:

$$\overline{Lv'} = l^2|\partial_y\bar{u}|, \tag{26a}$$

we obtain:

$$\overline{v'T'} = -l^2|\partial_y\bar{u}|\partial_y\bar{T}. \tag{26b}$$

It is reasonable to assume that the quantity $\overline{Lv'}$ and the mixing length L are the same for heat transfer as for momentum transfer, since the same mechanism is at work,[21] and it will be assumed that this is the case. Hence, we have:

$$\overline{u'T'} = \overline{v'T'} = -l^2|\partial_y\bar{u}|\partial_y\bar{T}, \tag{26c}$$

$$\overline{\rho'T'} = 0, \quad \overline{u'\rho'T'} = \overline{v'\rho'T'} = -l^2|\partial_y\bar{u}||\partial_y\bar{\rho}|\partial_y\bar{T}.$$

Since in this manner all the values are expressed in terms of $\bar{u}$, $\bar{v}$, $\bar{\rho}$, $\bar{T}$, we shall omit the bars above the letters denoting the mean velocity components, density, etc. Substitution of all the values of (25) and (26) into Eqs. (20) to (23) supplies the formulae:

two equations of motion:

$$\rho(u\partial_x u + v\partial_y u) + \partial_x p - (\partial_x\phi_1 + \partial_y\phi_2) = 0, \tag{27a}$$

$$\rho(u\partial_x v + v\partial_y v) + \partial_y p - (\partial_x\phi_2 + \partial_y\phi_3) = 0, \tag{27b}$$

$$\begin{aligned}\phi_1 &= 2l^2u\partial_y u\partial_y\rho + (l^2\rho + l^3\partial_y\rho)(\partial_y u)^2,\\ \phi_2 &= l^2(u+v)\partial_y u\partial_y\rho + (l^2\rho + l^3\partial_y\rho)(\partial_y u)^2,\\ \phi_3 &= 2l^2v\partial_y u\partial_y\rho + (l^2\rho + l^3\partial_y\rho)(\partial_y u)^2,\end{aligned} \tag{27c}$$

[18] Schlichting (*loc. cit.*) justifies Prandtl's assumptions. Namely, in a flow parallel to the x-axis with a velocity gradient in the transverse direction, the difference in velocities between two streamlines may be assumed to be equal $\partial_y\bar{u}$. A mass of fluid which moves transversely to the velocity $\bar{u}$ at the distance l will loose or gain the amount of velocity $l\partial_y\bar{u}$. Next assume that two masses of fluid with different velocities u' hit each other and because of that they deviate from their previous paths. The transverse velocity v' created in this manner may be assumed to be also proportional to the velocity u'. See also Goldstein, *loc. cit.*, p. 206.

[19] The stress τ should vary with $\partial_y\bar{u}$ which necessitates the using of the absolute value.

[20] Goldstein, p. 648.

[21] Goldstein, p. 649.

equation of continuity:

$$\partial_x(u\rho) + \partial_y(v\rho) = 0, \tag{28}$$

equation of energy:

$$p(\partial_x u + \partial_y v) + \overline{p'\partial_x u'} + \overline{p'\partial_y v'} + \mathcal{T}c_p[\rho(u\partial_x T + v\partial_y T) - (\partial_x\phi_4 + \partial_y\phi_5)] = 0, \tag{29}$$

$$\phi_4 = (l^2\partial_y u)\partial_y\rho\partial_y T + l^2\partial_y(\rho T)\partial_y u, \tag{29a}$$

$$\phi_5 = (l^2\partial_y u)\partial_y\rho\partial_y T + l^2\partial_y(\rho T)\partial_y u,$$

equation of state:

$$p = R(\rho T). \tag{30}$$

Below also the complete tensor forms (16c) (18) (19) will be used. For the two-dimensional flow they will be:

$$(\tau) = l^2\overline{\mathcal{T}}\rho\begin{pmatrix} 2\partial_x u & \partial_y u + \partial_x v \\ \partial_x v + \partial_y u & 2\partial_y v \end{pmatrix}, \tag{31}$$

$$\overline{\mathcal{T}}^2 = (\partial_y u + \partial_x v)^2 - 4\partial_x u\partial_y v, \tag{31a}$$

$$(\sigma) = l^2\begin{pmatrix} 2\partial_x(u^2)\partial_x\rho & \partial_x(uv)\partial_x\rho + \partial_y(uv)\partial_y\rho \\ \partial_y(uv)\partial_y\rho + \partial_x(uv)\partial_x\rho & 2\partial_y(v^2)\partial_y\rho \end{pmatrix}, \tag{32}$$

$$(\Pi) = l^2\begin{pmatrix} 2(\partial_x u)^2\partial_x\rho & \partial_x u\partial_x v\partial_x\rho + \partial_y u\partial_y v\partial_y\rho \\ \partial_x u\partial_y v\partial_y\rho + \partial_x u\partial_x v\partial_x\rho & 2(\partial_y v)^2\partial_y\rho \end{pmatrix}. \tag{33}$$

The two equations of motion will take the form:

$$\rho(u\partial_x u + v\partial_y u) + \partial_x\pi_{xx} + \partial_y\pi_{yx} = 0, \tag{34}$$

$$\rho(u\partial_x v + v\partial_y v) + \partial_x\pi_{xy} + \partial_y\pi_{yy} = 0, \tag{34a}$$

$$\begin{aligned} \pi_{xx} &= p - 2l^2[\overline{\mathcal{T}}\rho\partial_x u + 2u\partial_x u\partial_x\rho + l(\partial_x u)^2\partial_x\rho], \\ \pi_{yy} &= p - 2l^2[\overline{\mathcal{T}}\rho\partial_y v + 2v\partial_y v\partial_y\rho + l(\partial_y v)^2\partial_y\rho], \\ \pi_{yx} &= \pi_{xy} = -l^2[\overline{\mathcal{T}}\rho(\partial_y u + \partial_x v) + \partial_x(uv)\partial_x\rho + \partial_y(vu)\partial_y\rho \\ &\qquad + l(\partial_x u\partial_x v\partial_x\rho + \partial_y u\partial_y v\partial_y\rho)]. \end{aligned} \tag{34b}$$

V. THE FIRST APPROXIMATION.

1. *The Longitudinal Velocity Component.*

The symbols u_0, ρ_0, p_0, T_0, will refer to the undisturbed flow. Initially the first equation of motion will be taken into account. The symbols u, v, ρ, p, T, will denote the values in the disturbed flow, that is, in the wake. As the initial values let us take:

$$u = u_0, \qquad v = 0, \qquad p = p_0, \qquad \rho = \rho_0, \tag{35}$$

and as the first approximation:

$$u = u_0 - u_1, \qquad \partial_x u = -\partial_x u_1, \qquad \partial_y u = -\partial_y u_1. \tag{35a}$$

Following Prandtl, put forward the assumption that the mixing length is proportional to the width of the wake and that it is a function of x. Substituting

(35) and (35a) into (27a) leads to:

$$u_0\partial_x u_1 + 2l[l\partial_y u_1(\partial^2_{xy}u_1 + \partial^2_{yy}u_1) + \partial_x l(\partial_y u_1)^2] = 0. \tag{36}$$

Following Schlichting, put:

$$l \sim x^p, \qquad u_1 \sim u_0 x^{-q} f(\eta), \qquad \eta = x^{-p}y. \tag{37}$$

To calculate the values of p and q consider at first the momentum. The difference between the momentum values in front of the body and far behind the body must be constant and equal to the drag of the body. Hence for a unit breadth of the body we have:

$$\int_{-\infty}^{+\infty} \rho_0 u_0^2 dy - \int_{-\infty}^{+\infty} \rho u^2 dy = D = \text{const.} \tag{38}$$

With $\rho = \sim \rho_0$, $u\rho = \sim u_0\rho_0$, and with (35a) we obtain from (38):

$$\int_{-\infty}^{+\infty} u_1 dy = u_0 \int_{-\infty}^{+\infty} x^{-q} f(\eta) dy = \int_{-\infty}^{+\infty} x^{-q} x^p f(\eta) d\eta = \text{const.} \tag{39}$$

Since (39) must be valid for all values of x, it follows that $p = q$. Substituting the proportional value for u_1 from (37) into Eq. 36, and keeping only the terms of the highest order in x:

$$\begin{aligned} &\partial_x u_1 \sim x^{-q-1}, \qquad l^2\partial_y u_1 \partial^2_{yy} u_1 \sim x^{-2q-p}, \\ &l^2\partial_y u_1 \partial^2_{xy} u_1 \sim x^{-2q-1}, \qquad l\partial_x l(\partial_y u_1)^2 \sim x^{-2q-1}. \end{aligned} \tag{40}$$

Under the hypothesis that $q = p < 1$, the last two terms in (40) are of an order lower than the first two and we omit them in comparison with the first ones.[25] Since the remaining terms must be of equal order in x, we compare the exponents, obtaining the results:

$$-q - 1 = -2q - p, \qquad p = q = \tfrac{1}{2}. \tag{41}$$

Following previous authors let us introduce certain transformations:

$$x' = x/a, \qquad y' = y/a, \qquad a = c_D d, \qquad \eta = x'^{-\frac{1}{2}}y', \tag{42}$$

where the symbols denote:

c_D = experimental drag coefficient,
d = characteristic transverse dimension of the body.

Also we put:

$$l = cax'^{\frac{1}{2}} = ca^{-\frac{1}{2}}x^{\frac{1}{2}}, \qquad u_1 = u_0 x'^{-\frac{1}{2}} f(\eta). \tag{42a}$$

The dimension of a is inch, c is dimensionless. Both c and c_D must be determined from tests.

[25] Without any explanations Schlichting in all his calculations does not take into account the variation of l with x, that is, although $l = l(x)$ Schlichting excludes l in all the differentiation processes as if l were a constant. In the first approximation it does not play any part but in the approximations of higher order the number of terms is increased because of the fact that $l = l(x)$.

As we see in the first approximation we keep only the terms of the order $x'^{-\frac{1}{2}}$. In all other equations we shall keep also only the terms of this order in x' with the exception of the equation of state, in which the terms of the order x'^{-1} only will be kept. Hence, Eq. (36) will reduce to:

$$u_0\partial_x u_1 + 2l^2\partial_y u_1\partial^2_{yy}u_1 = 0. \tag{42b}$$

Keeping in mind that:

$$\partial_x\eta = -\tfrac{1}{2}x'^{-1}\eta, \qquad \partial_y\eta = x'^{-\frac{1}{2}}, \tag{42c}$$

we may easily obtain from (42b):[22]

$$f + \eta f' - 4c^2 f'f'' = 0, \quad \text{or} \quad d_\eta[\eta f - 2c^2 f'^2] = 0. \tag{42d}$$

Hence:

$$\eta f - 2c^2 f'^2 = \text{const.} \tag{43}$$

Because of the symmetry conditions we must have $f' = 0$ for $\eta = 0$, which means that the constant is zero. Let us put:

$$\eta = \eta_1(2c^2)^{\frac{1}{2}}, \tag{44}$$

and we obtain from (43):

$$f^{-\frac{1}{2}}df = \eta_1^{\frac{1}{2}}d\eta_1, \quad \text{or} \quad 2f^{\frac{1}{2}} = \tfrac{2}{3}\eta_1^{\frac{3}{2}} - 2A_1, \tag{45}$$

where $f = f(\eta_1)$ and $(-2A_1)$ is the constant of integration. With the transformations which we shall need in the subsequent calculations:

$$\eta_1 = \xi(3A_1)^{\frac{1}{3}}, \qquad z = \xi^{\frac{1}{3}}, \qquad A = A_1(2c^2)^{\frac{1}{3}}, \tag{46}$$

the result takes the following forms:

$$\begin{aligned} f(\eta_1) &= (\tfrac{1}{3}\eta_1^{\frac{3}{2}} - A_1)^2, & f(\eta) &= (2c^2)^{-1}(\tfrac{1}{3}\eta^{\frac{3}{2}} - A)^2, \\ f(\xi) &= A_1^2(\xi^{\frac{1}{3}} - 1)^2, & f(z) &= A_1^2(z - 1)^2. \end{aligned} \tag{47}$$

Throughout the entire calculations we shall have boundary conditions of two kinds. The boundary conditions of the first kind refer to the axis of symmetry (x − axis): $\partial_y u_1 = 0$ for $y' = 0$. This boundary condition is fulfilled. The boundary conditions of the second kind refer to the edge curve. In all the edge points we must have $u_1 = 0$ or $f(\eta_{e1}) = 0$ where the subscripts $e1$ denote the points of the first approximation edge curve. From (47) we have:

$$\eta_{e1} = (3A)^{\frac{1}{3}}, \quad \eta_{1e1} = (3A_1)^{\frac{1}{3}}, \quad \xi_{e1} = 1, \quad z_{e1} = 1, \quad y'_{e1} = x'^{\frac{1}{3}}(3A)^{\frac{1}{3}}. \tag{48}$$

This gives the first approximation of the edge curve. Since on the entire edge curve we must have a smooth transition of the additional velocity u_1 (from $u_1 \neq 0$ to $u_1 = 0$) we must have:

$$\partial_y u_1 = 0 \quad \text{for} \quad y' = y'_{e1}, \quad \text{or} \quad f'(\eta_1) = 0 \quad \text{for} \quad \eta_1 = \eta_{1e1}. \tag{48a}$$

This condition is fulfilled as one may easily verify. The breadth of the wake is

[22] The calculation of $f(\eta)$ follows the procedure given in Schlichting's paper. The subsequent calculations are different from those in Schlichting's paper because of the compressibility effect.

given by the formula:

$$2y'_{e1} = 2\frac{b}{a} = 2x'^{\frac{1}{2}}(3A)^{\frac{1}{2}} = 2(2c^2)^{\frac{1}{2}}(3A_1)^{\frac{1}{2}}x'^{\frac{1}{2}}. \tag{49}$$

The constant A_1 (or A) may be determined from momentum considerations. Namely, the drag per unit breadth of the obstacle is given by:

$$D = \int_{-y_{e1}}^{+y_{e1}} (\rho_0 u_0^2 - \rho u^2)dy. \tag{50}$$

Assuming in the first approximation $\rho_0 u_0 = \rho u$, we have:

$$2\rho_0 u_0 a \int_0^{y'_{e1}} u_1 dy' = \tfrac{1}{2} c_D d \rho_0 u_0^2. \tag{51}$$

Using (42) and (44) we transform (51):

$$(2c^2)^{\frac{1}{2}} \int_0^{\eta_{1e1}} (\tfrac{1}{3}\eta_1^{\frac{1}{2}} - A_1)^2 d\eta_1 = \tfrac{1}{4}, \tag{51a}$$

obtaining the final result:

$$(3A_1)^{\frac{1}{2}} A_1^2 = \frac{5}{9}(2c^2)^{-\frac{1}{2}}. \tag{52}$$

This equation enables one to find A_1.

2. The Transverse Velocity Component.[24]

Putting into the equation of continuity (28) $\rho = \rho_0$, $v = v_1$, and keeping only the terms of the order $x'^{-\frac{1}{2}}$ gives the expression:

$$\partial_{y'} v_1 - \partial_{x'} u_1 = 0. \tag{53}$$

A direct integration gives the result:

$$\begin{aligned} v_1(\eta) &= (-\tfrac{1}{2})x'^{-1}u_0 \eta f(\eta), & v_1(\eta_1) &= (-\tfrac{1}{2})(2c^2)^{\frac{1}{2}}x'^{-1}u_0\eta_1 f(\eta_1), \\ v_1(\xi) &= (-\tfrac{1}{2})(3A)^{\frac{1}{2}}x'^{-1}u_0\xi f(\xi), & v_1(z) &= (-\tfrac{1}{2})(3A)^{\frac{1}{2}}x'^{-1}u_0 z^{\frac{1}{2}} f(z). \end{aligned} \tag{54}$$

The boundary condition of the first kind: $v_1 = 0$ for $\eta = 0$ is fulfilled. Boundary conditions of the second kind: $v_1 = 0$ for $\eta = \eta_{e1}$, and $v_1' = 0$ for $\eta = \eta_{e1}$, are both fulfilled.

3. The Temperature.[25]

Substitute into the equation of energy (29) the values:

$$u = u_0 - u_1, \qquad v = v_1, \qquad \rho = \rho_0, \qquad p = p_0, \qquad T \sim x'^{-\frac{1}{2}}. \tag{55}$$

Keeping only the terms of the order $x'^{-\frac{1}{2}}$ gives the formula:

$$u_0 \partial_{x'} T + c^2 x' \partial_{y'}(\partial_{y'} T \partial_{y'} u_1) = 0. \tag{56}$$

Putting:

$$T = T_0 - T_1, \qquad T_1 = T_0 x'^{-\frac{1}{2}}\theta_1(\eta), \tag{57}$$

[24] Found in a manner entirely different from that of Schlichting.

[25] Not taken into account by Schlichting, who considered only an incompressible fluid.

and changing in (56) the independent variable into η transforms (56) into the expression:

$$-(\theta_1 + \eta\theta_1') + 2c^2(f'\theta_1'' + f''\theta_1') = 0. \tag{58}$$

This may be integrated directly and gives after changing into η_1:

$$-\eta_1\theta_1 + f'\theta_1' = \text{const.} \tag{59}$$

Since for $\eta_1 = 0$ we have also $f' = 0$, the constant is zero. Equation (59) is a linear differential equation. Elementary integration gives:

$$\theta_1 \exp(-2\beta) = \text{const.}, \qquad \beta = \log(\tfrac{1}{3}\eta_1^{\,3} - A_1), \tag{60}$$

or with the value for the constant equal to C':

$$\theta_1 = C'(\tfrac{1}{3}\eta_1^{\,3} - A_1)^2. \tag{61}$$

Hence the solution is:

$$\begin{aligned} T_1(\eta) &= C'x'^{-\frac{1}{3}}T_0 f(\eta), \qquad T_1(\eta_1) = C'x'^{-\frac{1}{3}}T_0 f(\eta_1), \\ T_1(\xi) &= C'x'^{-\frac{1}{3}}T_0 f(\xi), \qquad T_1(z) = C'x'^{-\frac{1}{3}}T_0 f(z). \end{aligned} \tag{62}$$

One may easily verify that all of the boundary conditions are fulfilled, that is:

$$T_1' = 0 \quad \text{for} \quad y' = 0, \quad T_1 = 0 \quad \text{for} \quad \eta = \eta_{e1}, \quad T_1' = 0 \quad \text{for} \quad \eta = \eta_{e1}.$$

The constant C' must be determined from test of T_1 for the points on the x – axis ($\eta = 0$).

4. The Pressure.[36]

Substituting into the second equation of motion (27b) the values:

$$u = u_0 - u_1, \qquad v = v_1, \qquad \rho = \rho_0, \qquad p = p_0 - p_1,$$

and keeping only the terms of the order $x'^{-\frac{1}{3}}$ gives the expression:

$$\partial_{y'}p_1 + c^2\rho_0 x'\partial_{y'}[(\partial_{y'}u_1)^2] = 0. \tag{63}$$

A direct integration gives:

$$p_1 + c^2x'^{-1}\rho_0 u_0^2 f'^2 = \text{const.} \tag{64}$$

Since for $\eta = \eta_{e1}$ we should have $p_1 = 0$ and since $f' = 0$ for $\eta = \eta_{e1}$, the constant is equal to zero. Simple transformations give:

$$p_1 = (-\tfrac{1}{3})x'^{-1}u_0^2\rho_0\eta f(\eta). \tag{65}$$

The pressure p_1 is zero for $\eta = 0$ and for $\eta = \eta_{e1}$. But p_1 should be an even function of η. Whereas $p_1' \sim (\eta f' + f)$ and it is not equal to zero for $\eta = 0$ since $f \neq 0$ for $\eta = 0$. Hence this is the degree of error in the first approximation. But the question arises whether in general p_1 represents the correct value since $p = (p_0 - p_1) > p_0$, which is impossible. We shall return to this question below. For the time being we will assume that $p_1 = 0$.

[36] Not performed by Schlichting.

5. The Density.

Substituting into the equation of state (30) the values:

$$T = T_0 - T_1, \qquad \rho = \rho_0 - \rho_1, \qquad p = p_0$$

and:

$$\rho_1 = \rho_0 x'^{-\frac{1}{2}} R_1(\eta), \tag{66}$$

and keeping only the terms of the order x'^{-1} gives the expression:[27]

$$\rho_1 T_1 = 0, \tag{67}$$

or

$$R_1 = 0. \tag{68}$$

Thus

$$\rho_1 = 0.^{28} \tag{69}$$

This is the solution.

6. Refinements.

In order to verify whether in reality the value of p_1, given by Eq. (65), should be taken into account, let us consider the second equation of motion with the total stress tensor forms according to Prandtl and the author's suggestions. Substituting into (34a) the values: $u = u_0 - u_1$, $v = v_1$, $p = p_0 - p_1$, $\rho = \rho_0$, and keeping only the terms of the order $x'^{-\frac{1}{2}}$, we obtain $\partial_{y'} p_1 = 0$. As concerning the order of $\overline{T}$, keeping only the terms of the highest order in x', we may easily verify that $\overline{T} \sim x'^{-1}$. Hence $p_1 = \text{const.}$ in each cross section. But from the boundary conditions $p_1 = 0$. We see that the total stress tensor form gives a better approximation to p_1 and the assumption, previously made, that $p_1 = 0$, is fully justified.

VI. THE SECOND APPROXIMATION.

1. The Longitudinal Velocity Component.

Let us put into the first equation of motion (27a):

$$u = u_0 - u_1 - u_2, \quad u_2 = u_0 x'^{-1} g(\eta), \quad v = v_1, \quad p = p_0, \quad \rho = \rho_0, \tag{70}$$

Keeping only the terms of the order x'^{-2} we obtain:[29]

$$u_1 \partial_{x'} u_1 - u_0 \partial_{x'} u_2 - v_1 \partial_{y'} u_1 - c^2 (\partial_{y'} u_1)^2 - 2c^2 [x' (\partial_{y'} u_1 \partial^2_{x'y'} u_1 + \partial_{y'} u_1 \partial^2_{y'y'} u_2 + \partial_{y'} u_2 \partial^2_{y'y'} u_1) + 2c_1 x'^{\frac{1}{2}} \partial_{y'} u_1 \partial^2_{y'y'} u_1] = 0. \tag{71}$$

Changing to η gives:

$$g + \tfrac{1}{2}\eta g' - \tfrac{1}{2} f^2 - 2c^2 [(g'' - \tfrac{1}{2} f') f' + (g' - \tfrac{1}{2}\eta f' + 2c_1 f') f''] = 0. \tag{72}$$

[27] Keeping only the terms of the order $x'^{(-\frac{1}{2})}$ supplies the expression which might be considered as an initial approximation:

$$\rho_1 = - C' x'^{(-\frac{1}{2})} \rho_0 f.$$

But this would mean that the density in the wake $\rho > \rho_0$, which is not probable. Hence we somit this solution.

[28] The result obtained above, that is, $\rho_1 = 0$ is not in a contradiction to the formula $p_1 = 0$.

[29] The term $\partial_{y'} u_1 \partial^2_{x'y'} u_1$ without explanation was not taken into account by Schlichting. Also he omitted the term $c^2 (\partial_{y'} u_1)^2$ due to the variation of l with x'.

Changing to η_1 gives:

$$g_1 - (\tfrac{1}{8}\eta_1 - \tfrac{1}{2}A_1\eta_1^{-3})g_1' - (\tfrac{1}{8}\eta_1^2 - A_1\eta_1^{-2})g_1'' - \tfrac{1}{2}(\tfrac{1}{8}\eta_1^4 - A_1)^4 + (2c^2)^{-1}(\tfrac{1}{8}\eta_1^3 - \tfrac{3}{2}A_1\eta_1^{-1} + \tfrac{3}{4}A_1^3) - c_1(\tfrac{1}{8}\eta_1^3 - \tfrac{3}{2}A_1\eta_1^{-1} + A_1^3) = 0. \quad (73)$$

In (73) the subscript 1 in $g_1(\eta_1)$ denotes the function of η_1. Changing to z gives:

$$\tfrac{1}{4}z(1-z)d_{zz}^2g_1 + \tfrac{1}{2}(1-z)d_zg_1 + g_1(z) - \tfrac{1}{2}A_1^4(z-1)^4 - A_1^3c_1(z-1)(4z-1) + \tfrac{1}{4}A_1^2(3A)^{-1}z^{-1}(6z^2 - 9z + 3) = 0. \quad (74)^{30}$$

Initially considering the homogeneous part of this equation:

$$z(1-z)d_{zz}^2g_1 + \tfrac{3}{2}(1-z)d_zg_1 + \tfrac{3}{4}g_1 = 0. \quad (75)$$

This is a Gauss equation. Its general type is:

$$z(1-z)y'' + [\gamma - (\alpha + \beta + 1)z]y' - \alpha\beta y = 0. \quad (76)$$

In our case: $\gamma = \frac{3}{2}$, $\alpha_1 = 1$, $\beta_1 = -\frac{1}{2}$, or $\alpha_2 = -\frac{1}{2}$, $\beta_2 = 1$. The two linearly independent solutions of (75) in the unit circle around the origin are:[31]

$$g_1 = C_1F(\alpha, \beta, \gamma, z) + C_2z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z), \quad (77)$$

where the known hypergeometric series is:

$$F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1!\gamma}z + \frac{\alpha(\alpha+1)}{2!}\frac{\beta(\beta+1)}{\gamma(\gamma+1)}z^2 + \cdots \quad (78)$$

The constants C_1 and C_2 will be determined from the boundary conditions. The function g_1 (or g) must be an even function of η or z because of symmetry conditions. That means that $g_1' = 0$ for $z = 0$. Hence after substituting the numerical values for α, β, γ we obtain:

$$g_1'(z) = -4C_1 + \cdots + \tfrac{3}{2}C_2z^{-\frac{1}{2}} + \tfrac{5}{8}C_2z^{\frac{1}{2}} + \cdots \quad (79)$$

For $z = 0$, the first term with $C_2 \to \infty$ and this term is not balanced by any other term. Hence C_2 must be zero. But that also requires that $C_1 = 0$.[32] Hence both solutions drop out. We assume that both solutions of the homogeneous equation for the whole plane drop out. We have to find the particular integrals of the nonhomogeneous equation. To this end let us split Eq. (74) into three separate equations and solve each of them independently. The subscripts 2, 3, 4 will be used:

$$\begin{aligned} L(g_2) &= -\tfrac{1}{2}A_1^4(z-1)^4 = 0, \\ L(g_3) &= -A_1^3c_1(z-1)(4z-1) = 0, \\ L(g_4) &= \tfrac{1}{4}A_1^2(3A)^{-1}z^{-1}(6z^2 - 9z + 3) = 0, \\ L(g_i) &= \tfrac{3}{4}z(1-z)d_{zz}^2g_i + \tfrac{1}{2}(1-z)d_zg_i + g_i. \end{aligned} \quad (80)$$

[30] Of course, this equation is different from Schlichting's equation.

[31] Heinrich Weber, *Die Partiallen Differential-Gleichungen der Mathematischen Physik*, Braunschweig, Verlag von F. Vieweg und Sohn, 1901.

[32] This proof is much simpler than that of Schlichting.

To find g_2 let us represent it by a polynomial of the fourth degree in z with, for the time being, unknown coefficients. Substitution into the first equation of (80) and comparison of the terms containing equal powers in z gives the result:

$$g_2(z) = A_1^4(-0.05z^4 + 0.29z^3 - 0.84z^2 + 0.2z + 0.4), \tag{81}$$

$$g_2(\xi) = A_1^4(-0.05\xi^6 + 0.29\xi^5 - 0.84\xi^3 + 0.2\xi^2 + 0.4). \tag{81a}$$

As we see for $\xi = \eta = 0$, we have $g_2' = 0$. For $\xi = \xi_{e1}$ also $g_2 = 0$. But for this value of ξ we have $g_2' = -(1.215A_1^4)$. This is the error of the second approximation. Similarly g_3 was solved:

$$g_3(z) = A_1^3c_1(-\tfrac{8}{3}z^2 + \tfrac{10}{3}z - \tfrac{2}{3}), \tag{82}$$

$$g_3(\xi) = A_1^3c_1(-\tfrac{8}{3}\xi^3 + \tfrac{10}{3}\xi^2 - \tfrac{2}{3}). \tag{82a}$$

As we see, $g_3' = 0$ for $\xi = \eta = 0$, $g_3 = 0$ for $\xi = \xi_{e1}$, $g_3' = (-3)A_1^3c_1$ for $\xi = \xi_{e1}$. Besides that Schlichting noticed that $\int_0^1 g_3 d\xi = 0$, which means that the function g_3 and the coefficient c_1 have no contribution to the total value of the momentum behind the body.

To find g_4 [33] for the unit circle around the origin put $z = t^2$, and represent g_4 by a series in increasing powers in t. Substitution into (80) and return to the variable z gives the solution (the symbol g_{41} is used):

$$g_{41}(z) = a_0 + a_1z^{\frac{1}{2}} + a_2z + a_4z^{4/3} + a_5z^{5/3} + a_6z^2 + \cdots, \tag{83}$$

$$g_{41}(\xi) = a_0 + a_1\xi^1 + a_2\xi^2 + a_4\xi^{\frac{8}{3}} + a_5\xi^{5/3} + a_6\xi^3 + \cdots,^{34} \tag{83a}$$

From the boundary conditions we have $g_{41}' = 0$ for $\xi = 0$ or $\eta_1 = 0$, which necessarily means that $a_1 = 0$ and also $a_4 = 0$. The determination of a_0 will be given below. To find g_{42} for the unit circle around $z = 1$, let us put $z = r + 1$, expand $z^{\frac{1}{2}} = (1 + r)^{\frac{1}{2}}$ by the binomial theorem and let us express g_{42} by a power series in r. The result is:

$$g_{42} = c_1'(z - 1) + c_2(z - 1)^2 + c_3(z - 1)^3 + \cdots, \tag{84}$$

$$g_{42} = c_1'(\xi^2 - 1) + c_2(\xi^2 - 1)^2 + c_3(\xi^2 - 1)^3 + \cdots \tag{84a}$$

From the boundary conditions we have $g_{42} = 0$ for $\xi_{e1} = 1$, which is fulfilled. Also $g_{42}' = 0$ for $\xi_{e1} = 1$, which means that $c_1' = 0$. Other coefficients are given in the Appendix.

We are able now to determine the constant a_0 in (83) by application of the momentum considerations. Namely, substituting into (50) and (51) the value $(u_0 - u) = u_1 + u_2$ we obtain:

$$\int_0^{y'_{e1}} (u_1 + u_2)dy' = \tfrac{1}{4}u_0 \tag{85}$$

[33] This function was not taken into account by Schlichting.

[34] The values of the coefficients are given in the Appendix at the end of the paper, to appear in the February issue of this JOURNAL.

or

$$x'^{\frac{1}{2}}(3A)^{\frac{1}{3}}\int_0^1 (u_1 + u_2)d\xi = \tfrac{1}{4}u_0. \tag{85a}$$

Calculating:

$$\int_0^1 u_1 d\xi = \frac{9}{20}A^2(2c^2)^{-1}x'^{-\frac{1}{2}}u_0,$$

$$\int_0^1 u_2 d\xi = u_0 x'^{-1}\int_0^1 (g_2 + g_3 + g_4)d\xi,$$

$$\int_0^1 g_2 d\xi = \frac{24.3}{77}A_1^4, \qquad \int_0^1 g_3 d\xi = 0,$$

making an arbitrary assumption as concerning the integration of the function g_4:

$$\int_0^1 g_4 d\xi = \int_0^{0.5} g_{41}d\xi + \int_{0.5}^1 g_{42}d\xi = 0.5a_0 + C(g_4),$$

and substituting all the results in (85a) gives:

$$a_0 = \tfrac{1}{3}x'^{\frac{1}{2}}\left[(3A)^{-\frac{1}{3}} - \tfrac{8}{3}A_1^2\right] - \frac{48.6}{77}A_1^4 - 2C(g_4), \tag{86}$$

a_0 is a constant with reference to z or ξ but is a function of x'. The final result is:

$$u_2 = u_0 x'^{-1}(g_2 + g_3 + g_{4i}), \qquad i = 1,2. \tag{87}$$

2. Second Approximation to the Edge Curve.

To find the second approximation to the edge curve, which will not coincide with the first approximation edge curve, we put:

$$\partial_{y'}u_1 + \partial_{y'}u_2 = 0. \tag{88}$$

The points, satisfying Eq. (88) up to a certain order of approximation, will lie on the required edge curve. We obtain:

$$\partial_{y'}u_1 = 3A_1^2(3A)^{-\frac{1}{3}}x'^{-1}u_0(\xi^2 - \xi^{\frac{1}{2}}), \qquad (\xi^2 - \xi^{\frac{1}{2}}) = f'(\xi). \tag{88a}$$

Expand the function of ξ in Taylor series around $\xi = 1$ and keep only the first non-vanishing term:

$$f'(\xi) = f'(1) + \frac{f''(1)(\xi - 1)}{1!} + \cdots = \tfrac{3}{2}(\xi - 1) + \cdots, \text{ since } f'(1) = 0.$$

Hence:

$$\partial_{y'}u_1 = 4.5A_1^2(3A)^{-\frac{1}{3}}x'^{-1}u_0(\xi - 1) + \cdots. \tag{88b}$$

In the same way:

$$\partial_{y'}u_2 = (3A)^{-\frac{1}{3}}x'^{-1}u_0(d_2g_2 + d_2g_3 + d_2g_{42}), \tag{89}$$

$$g_2' = -1.215A_1^4, \tag{89a}$$

by expansion in Taylor series with $g_3'(1) \neq 0$.

$$g_3' = -3A_1^2c_1, \tag{89b}$$

from Taylor series expansion with $g_3'(1) \neq 0$.

$$g_{42}' = \tfrac{3}{2}c_2(\xi - 1) + \cdots, \quad \text{with} \quad g_{42}'(1) = 0, \qquad g_{42}''(1) \neq 0. \tag{89c}$$

Substituting (88b), (89 a to c) into (87) gives the result:[35]

$$\xi = \xi_{e2} = 1 + Kx'^{-\frac{1}{3}}, \quad K = A_1^2(1.215A_1^2 + 3c_1)[4.5(A_1^2 + c_2)]^{-1}, \tag{90}$$

$$y_{e2}' = (3A)^{\frac{1}{3}}(x'^{\frac{1}{3}} + K). \tag{90a}$$

That the other boundary conditions of the second kind, that is $(u_1 + u_2)$ should vanish for $\xi = \xi_{e2}$, is not perfectly fulfilled and we have to find the order of the first non-vanishing terms. Expanding $(u_1 + u_2)$ in Taylor series and keeping in mind that $f(1) = f'(1) = g_3(1) = g_3'(1) = g_{42}(1) = g_{42}'(1) = 0$, we obtain:

$$u_1(\xi_{e2}) = \tfrac{2}{3}A_1^2x'^{-\frac{1}{3}}u_0K^2 + \cdots, \tag{91}$$

$$u_2(\xi_{e2}) = -A_1^2x'^{-\frac{1}{3}}u_0K(1.215A_1^2 + 3c_1) + \cdots \tag{91a}$$

The order $x'^{-\frac{1}{3}}$ is acceptable.[36]

3. The Transverse Velocity Component.

Putting into the equation of continuity (28):

$$u = u_0 - u_1 - u_2, \qquad v = v_1 + v_2, \qquad \rho = \rho_0, \qquad v_2 \sim x'^{-\frac{1}{3}},$$

and keeping only the terms of the order x'^{-2} gives the expression:

$$\partial_{y'}v_2 - \partial_{x'}u_2 = 0. \tag{92}$$

Changing to z and keeping in mind that:

$$x' = x'(z), \qquad d_zx' = -\tfrac{1}{3}z^{-1}x', \qquad d_z(x'^{-1}) = \tfrac{1}{3}x'^{-1}z^{-1}, \tag{93}$$

gives the formula:

$$v_2 = -\tfrac{1}{2}(3A)^{\frac{1}{3}}x'^{-\frac{1}{3}}u_0\left[\frac{4}{3}\int z^{-\frac{1}{3}}(g_2 + g_3 + g_{41})dz + \int z^{\frac{1}{3}}(g_2' + g_3' + g_{41}')dz\right] + C_3, \tag{94}$$

$$\int z^{-\frac{1}{3}}g_2dz = A_1^4\left(-\frac{0.15}{14}z^{14/3} + \frac{0.87}{11}z^{11/3} - 0.315z^{8/3} + 0.12z^{5/3} + 0.6z^{\frac{1}{3}}\right). \tag{94a}$$

[35] This result is different from Schlichting's result.

[36] This result is in full accordance with Schlichting's result, because the contribution of the function g_{43}, omitted by Schlichting, is of an order lower than $x'^{-\frac{1}{3}}$. Only the constant K is different (see 90).

This integral is equal to zero for $\xi = 0 (z = \xi^{\frac{1}{3}})$.

$$\int z^{-\frac{1}{3}} g_3 dz = A_1^2 c_1 (- z^{8/3} + 2z^{5/3} - z^{\frac{1}{3}}). \tag{94b}$$

This integral is also equal to zero for $\xi = 0$. For the unit circle around the origin we have:

$$\int z^{-\frac{1}{3}} g_{41} dz = 1.5 a_0 z^{\frac{1}{3}} + 0.6 a_2 z^{5/3} + \tfrac{3}{7} a_5 z^{7/3} + \tfrac{3}{8} a_6 z^{8/3} + \cdots \tag{94c}$$

This series vanishes for $\xi = 0$. To obtain the series expansion around the point $z = 1$, let us put in (84) $(z - 1) = r$, expand $z^{-\frac{1}{3}} = (1 + r)^{-\frac{1}{3}}$ by the binomial theorem, integrate term by term and return to z. The result is:

$$\int z^{-\frac{1}{3}} g_{42} dz = \tfrac{1}{3} c_2 (z - 1)^3 + \tfrac{1}{4}(-\tfrac{1}{3} c_2 + c_3)(z - 1)^4 + \tfrac{1}{5}(\tfrac{2}{9} c_2 - \tfrac{1}{3} c_3 + c_4)(z - 1)^5 + \cdots \tag{94d}$$

In a similar way:

$$\int z^{\frac{1}{3}} g_2' dz = A_1^4 \left(- \frac{0.3}{7} z^{14/3} + \frac{2.61}{11} z^{11/3} - 0.63 z^{8/3} + 0.12 z^{5/3} \right). \tag{95}$$

This function vanishes for $z = \xi = 0$.

$$\int z^{\frac{1}{3}} g_3' dz = 2A_1^2 c_1 (- z^{8/3} + z^{5/3}), \tag{95a}$$

$$\int z^{\frac{1}{3}} g_{41}' dz = \tfrac{3}{8} a_3 z^{5/3} + \tfrac{3}{7} a_5 z^{7/3} + \tfrac{3}{4} a_6 z^{8/3} + \cdots. \tag{95b}$$

The last two functions vanish for $z = \xi = 0$.

$$\int z^{\frac{1}{3}} g_{42}' dz = c_2 (z - 1)^3 + (\tfrac{4}{3} c_2 + c_3)(z - 1)^3 + (-\tfrac{1}{15} c_2 + 0.5 c_3 + c_4)(z - 1)^4 + (\tfrac{8}{405} c_2 - \tfrac{1}{15} c_3 + \tfrac{3}{15} c_4 + c_5)(z - 1)^5 + \cdots. \tag{95c}$$

Summarizing we obtain the result:

$$v_2 = (- x'^{-\frac{1}{3}}) u_0 G + C_3, \quad G = G_2 + G_3 + G_{41}^2, \quad i = 1, 2. \tag{96}$$

The function v_2 must be an odd function, i.e. for $y' = 0$ we must have $v_2 = 0$. This means that $C_3 = 0$ and the boundary conditions of the first kind are fulfilled. The boundary conditions of the second kind require that $v_1 + v_2 = 0$ for $\xi = \xi_{s2}$. Let us find the order of the first nonvanishing terms by expansion in Taylor series around $\xi = 1$:

$$v_1(\xi_{s2}) = (- \tfrac{2}{3}) A_1^2 (3A)^{\frac{1}{3}} x'^{-2} u_0 K^2 + \cdots,$$

$$v_1(1) = v_1'(1) = 0, \qquad v_1''(1) \neq 0, \tag{97}$$

$$v_2(\xi_{s2}) = (- x'^{-\frac{1}{3}}) u_0 G(\xi_{s2}).^{****} \tag{97a}$$

**** Appendix.

The first non-vanishing term is of the acceptable order $x'^{-\frac{1}{2}}$. The last boundary condition is that $v_1'(y') + v_2'(y') = 0$ for $y' = y'_{s2}$. By expansions in Taylor series the first non-vanishing terms are the following: (the symbol f denotes always the function of ξ entering the expression considered):

$$\partial_{y'}v_1 = (-\tfrac{1}{2})A_1^2x'^{-\frac{1}{2}}u_0(4\xi^3 - 5\xi^{\frac{1}{2}} + 1)$$
$$= (-\tfrac{1}{2})A_1^2x'^{-2}u_0K + \cdots, \qquad f(1) = 0, \qquad f'(1) \neq 0, \quad (98)$$
$$\partial_{y'}v_2 = (-\tfrac{3}{4})x'^{-2}u_0(\bar{G}_2 + \bar{G}_3 + \bar{G}_{42}).^{****} \quad (98a)$$

Hence the first non-vanishing terms are of the order x'^{-2}.

4. The Temperature.

Substitute into the equation of energy (29) the values:

$$u = u_0 - u_1 - u_2, \qquad v = v_1 + v_2, \qquad \rho = \rho_0, \qquad p = p_0, \quad (99)$$
$$T = T_0 - T_1 - T_2, \qquad T_2 = T_0x'^{-1}\theta_2(\eta)$$

and keep only the terms of the order x'^{-2}. Using Eq. (92) we obtain from (29):

$$u_1\partial_{x'}T_1 - u_0\partial_{x'}T_2 - v_1\partial_{y'}T_1 - c^2x'(\partial^2_{y'y'}u_1\partial_{y'}T_1 + \partial_{y'}u_1\partial^2_{y'y'}T_1$$
$$+ \partial^2_{y'y'}u_1\partial_{y'}T_2 + \partial_{y'}u_1\partial^2_{y'y'}T_2 + \partial^2_{y'y'}u_2\partial_{y'}T_1 + \partial_{y'}u_2\partial^2_{y'y'}T_1) \quad (100)$$
$$- c^2\partial_{y'}u_1\partial_{y'}T_1 - 2c_1c^2x'^{\frac{1}{2}}(\partial^2_{y'y'}u_1\partial_{y'}T_1 + \partial_{y'}u_1\partial^2_{y'y'}T_1) = 0.$$

Introduce the variable η:

$$2(\theta_2 + \eta\theta_2') - f\theta_1 + c^2[(2\theta_1' + \eta\theta_1'' - 2\theta_2'')f' + (\eta\theta_1' - 2\theta_2')f''$$
$$- 2(g''\theta_1' + g'\theta_1'')] - 4c_1c^2(f''\theta_1' + f'\theta_1'') = 0. \quad (101)$$

Introduce the variable η_1:

$$f'\theta_2'' + (f'' - 2\eta_1)\theta_2' - 2\theta_2 + f\theta_1 - (2c^2)^{-1}c^2[2f'\theta_1' + \eta_1(f''\theta_1' + f'\theta_1'')]$$
$$+ g_1'\theta_1'' + g_1''\theta_1' + 2c_1(f''\theta_1' + f'\theta_1'') = 0. \quad (102)$$

Introduce the variable z:

$$3A_1^{-2}zf'\theta_2'' + [A_1^{-2}(2f' + 3zf'') - 24z]\theta_2' - 16\theta_2 + 8C'f^2$$
$$- 3(3A)^{\frac{1}{2}}A_1^{-2}C'z^{\frac{1}{2}}(f'^2 + zf'f'')$$
$$+ A_1^{-2}C'[2g'f' + 3z(f'g'' + f''g')]$$
$$+ 4A_1^{-2}C'c_1(f'^2 + 3zf'f'') = 0. \quad (103)$$

In Eq. (103) the substitution was used: $\theta_1 = C'f$. Take at first the homogeneous equation and after introducing the function $f(z)$ obtain:

$$3z(z - 1)\theta_2'' - (7z + 2)\theta_2' - 8\theta_2 = 0. \quad (104)$$

Representing θ_2 by a polynomial in z we easily obtain:

$$\theta_2 = k_0(z - 1)^4 = k_0(\xi^{\frac{1}{2}} - 1)^4, \quad (104a)$$

where k_0 is a constant. But the boundary condition is that $\theta_2 = 0$ for $\xi = \xi_{s2}$ which is only possible if $k_0 = \theta_2 = 0$. To find the particular solutions of the

non-homogeneous equation it is necessary to solve the following set of equations:

$$L(\theta_{21}) + 2C'f^2 = 0, \tag{105a}$$

$$L(\theta_{22}) + 2C'A_1^{-2}c_1f'^2 = 0, \tag{105b}$$

$$L(\theta_{23}) + 6C'A_1^{-2}c_1zf'f'' = 0, \tag{105c}$$

$$L(\theta_{24}) - \tfrac{3}{2}C'(3A)^{\frac{1}{3}}A_1^{-2}z^{\frac{1}{3}}(f'^2 + zf'f'') = 0, \tag{105d}$$

$$L(\theta_{25}) + \tfrac{1}{2}C'A_1^{-2}[2f'g' + 3z(f'g'' + f''g')] = 0. \tag{105e}$$

$L(\theta_{2i})$, $i = 1$ to 5, is given by the left hand side of Eq. (104). Below each of Eqs. (105) will be solved separately. The symbol B.C. will denote "boundary conditions." All the equations were solved by expansion in polynomials or power series in z.

(*a*) θ_{21}.

For the unit circle around the origin obtain:

$$\theta_{21} = m_0 + m_1z + m_2z^2 + m_3z^3 + \cdots. \tag{106}$$

The coefficient m_0 gives the distribution of θ_{21} along the x-axis and must be determined from tests. Since $\theta'_{21} = 0$ for $\eta_1 = 0$ the B.C. is fulfilled. Changing z to r by $(z - 1) = r$, we may easily find the expansion around $z = 1$:

$$\bar{\theta}_{21} = n_4(z - 1)^4 + n_5(z - 1)^5 + n_6(z - 1)^6 + \cdots. \tag{107}$$

For $z = 1$, $\bar{\theta}_{21} = 0$. For $\xi = \xi_{s2}$, we put:

$$T_1(\xi_{s2}) + T_2(\xi_{s2}) = 0. \tag{108}$$

We may easily find:

$$T_1(\xi_{s2}) = x'^{-\frac{1}{3}}T_0A_1^2[t(x')]^2, \tag{109a}$$

$$t(x') = [(1 + Kx'^{-\frac{1}{3}})^{\frac{1}{3}} - 1] = \tfrac{1}{3}Kx'^{-\frac{1}{3}} + \tfrac{1}{9}K^2x'^{-1} - \tfrac{1}{16}K^3x'^{-\frac{1}{3}} + \cdots. \tag{109b}$$

The function $t(x')$ is a series in descending powers in x'. The condition (108) will give the value of n_4:

$$n_4 = -\{A_1^2x'^{\frac{1}{3}}C'[t(x')]^{-2} + n_5[t(x')] + n_6[t(x')]^2 + \cdots\}. \tag{110}$$

The last boundary condition is that $T_1'(y') + T_2'(y') = 0$ for $y' = y'_{s2}$. Expanding in Taylor series around $\xi = 1$ we obtain:[37]

$$T_1'(y'_{s2}) = 3T_0(3A)^{-\frac{1}{3}}A_1^2x'^{-\frac{1}{3}}(\xi_{s2}^2 - \xi_{s2}^{\frac{1}{3}}) = 3T_0(3A)^{-\frac{1}{3}}A_1^2x'^{-\frac{1}{3}}F(\xi_{s2}). \tag{111}$$

Expanding the function $F(\xi_{s2})$ in Taylor series around $\xi = 1$, we obtain:

$$F(\xi_{s2}) = \tfrac{1}{3}Kx'^{-\frac{1}{3}} + \cdots, \qquad \text{with} \qquad F(1) = 0, \qquad F'(1) = \tfrac{1}{3}, \tag{111a}$$

[37] In all such calculations we assume that the function $t(x') = [(1 + Kx'^{-\frac{1}{3}})^{\frac{1}{3}} - 1] = (\xi_{s2}^{\frac{1}{3}} - 1) = \tau(\xi_{s2})$ is not subject to variations with variations of the independant variable ξ, i.e., the values ξ_{s2} are assumed to be fixed values.

Hence the first non-vanishing term in $T_1'(y'_{*2})$ is of the order x'^{-1}. Similarly:

$$T_2'(y'_{*2}) = (3A)^{-\frac{1}{3}} T_0 x'^{-\frac{1}{3}} \sum_{i=1}^{i=5} \bar{\theta}'_{2i}(\xi_{*2}), \tag{112}$$

$$\bar{\theta}'_{21}(\xi_{*2}) = \frac{81}{4} n_4 K^3 x'^{-\frac{1}{3}} + \cdots,$$
$$\bar{\theta}'_{21}(1) = \bar{\theta}''_{21}(1) = \bar{\theta}'''_{21}(1) = 0, \qquad \bar{\theta}^{IV}_{21}(1) \neq 0. \tag{112a}$$

But one may easily verify that $n_4 \sim x'^{\frac{1}{3}} + \cdots$. Hence the first non-vanishing term in $T_2'(y'_{*2})$ is of the order $x'^{-\frac{1}{3}}$. In a similar way the other particular solutions were found. The bar over the θ means that the solution refers to the unit circle around $z = 1$. The results are:

(*b*) θ_{22}.

$$\theta_{22} = \bar{p}_0 + \bar{p}_1 z + \bar{p}_2 z^2 + \bar{p}_3 z^3 + \bar{p}_4 z^4. \tag{113}$$

$$\text{B.C. } \theta'_{22}(\eta_1)\big|_{\eta_1=0} = 0, \qquad \theta_{22}(\xi_{*2}) = 0. \tag{114}$$

This condition gives the value for $\bar{p}_0$. The first non-vanishing term is of the order $x'^{-\frac{1}{3}}$ (see Eq. 112).

(*c*) θ_{23}.

$$\theta_{23} = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4. \tag{115}$$

$$\text{B.C. } \theta'_{23}(\eta_1) = 0 \quad \text{for} \quad \eta_1 = 0, \qquad \theta_{23}(\xi_{*2}) = 0. \tag{116}$$

This condition gives the value of s_0. The first non-vanishing term is of the order $x'^{-\frac{1}{3}}$ (see Eq. 112).

(*d*) θ_{24}.

Put $z = t^3$ and solve. Keeping in mind that

$$\frac{d\theta_{24}}{dz} = (\tfrac{1}{3}) t^{-2} \theta'_{24}(t), \qquad \frac{d^2\theta_{24}}{dz^2} = (\tfrac{1}{3}) t^{-4} [\theta''_{24}(t) - 2t^{-1}\theta'_{24}(t)], \tag{117}$$

and returning to z, one obtains:

$$\theta_{24} = E_0 + E_1 z + E_{\frac{5}{3}} z^{5/3} + E_2 z^2 + E_{\frac{8}{3}} z^{8/3} + \cdots \tag{118}$$

E_0 from tests.

$$\text{B.C. } \theta'_{24}(\eta_1)\big|_{\eta_1=0} = 0. \tag{119}$$

In the solution of $\bar{\theta}_{24}$ put $(z - 1) = r$, expand $z^{\frac{1}{3}}$ by the binomial theorem:

$$\bar{\theta}_{24} = F_0 + F_1(z-1) + F_2(z-1)^2 + F_3(z-1)^3 + \cdots. \tag{120}$$

F_4 cannot be determined from the series expansion and must be defined from the condition that

$$\bar{\theta}_{24}(\xi)\big|_{\xi=\xi_{*2}} = 0. \tag{121}$$

(*e*) θ_{25}.

$$\xi_{25} = N_0 + N_1 z + N_{\frac{5}{3}} z^{5/3} + N_2 z^2 + N_{\frac{8}{3}} z^{8/3} + N_3 z^3 + \cdots. \tag{122}$$

N_0 must be determined from tests. B.C. $\theta'_{25}(\eta_1)\big|_{\eta_1=0} = 0$ is fulfilled.

$$\bar{\theta}_{25} = L_0 + L_1(z-1) + L_2(z-1)^2 + L_3(z-1)^3 + \cdots \tag{123}$$

L_4 cannot be determined from the series expansion and must be defined from the condition that $\bar{\theta}_{25}(\xi_{*2}) = 0$.

5. The Pressure.

Let us put into the second equation of motion (27b) the values:

$$u = u_0 - u_1 - u_2, \qquad v = v_1 + v_2, \qquad \rho = \rho_0, \qquad p = p_0 - p_2, \quad (124)$$
$$p_2 = p_0 x'^{-\frac{1}{2}} \Pi_2,$$

and keep only the terms of the order x'^{-2}. We obtain:

$$\partial_{y'} p_2 = \rho_0 \{ u_0 \partial_{x'} v_1 - 2c^2 [x' (\partial_{y'} u_1 \partial^2_{x'y'} u_1 + \partial_{y'} u_1 \partial^2_{y'y'} u_2 + \partial^2_{y'y'} u_1 \partial_{y'} u_2)$$
$$+ 2c_1 x'^{\frac{1}{2}} \partial_{y'} u_1 \partial^2_{y'y'} u_1] - c^3 (\partial_{y'} u_1)^2 \}, \quad (125)$$

Changing to z we have:

$$p_2' (u_0^2 \rho_0)^{-1} x'^{\frac{1}{2}} = \tfrac{1}{8} (3A)^{4/3} A_1 z^{\frac{1}{2}} (z-1)(2z-1)$$
$$+ \tfrac{3}{4} (3A)^{-\frac{1}{3}} c^2 z^{\frac{1}{2}} f' (f' + 3zf'')$$
$$- \tfrac{1}{6} (3A)^{-\frac{1}{3}} A^{-3} c^2 z^{-\frac{1}{2}} \{ 2f' (g_2' + g_3' + g_{41}')$$
$$+ 3z [f' (g_2'' + g_3'' + g_{41}'') + f'' (g_2' + g_3' + g_{41}')]$$
$$+ 2c_1 (f'^2 + 3zf' f'') \}. \quad (126)$$

Substituting all the functions f and g in (126), ordering according to ascending powers of z and integrating directly term by term gives the solution for the unit circle around $z = 0$ in the form:

$$\Pi_2 = u_0^2 \rho_0 p_0^{-1} (L' + \tfrac{2}{3} L_2' z^{\frac{1}{2}} + \tfrac{3}{4} L_4' z^{4/3} + \tfrac{3}{5} L_5' z^{5/3} + \cdots), \quad (127)$$

where the constant coefficients are given in the Appendix. The coefficient L' which gives the value of the pressure on the x-axis must be taken from tests. $\Pi_2' = (\frac{3}{2}) L_2'$ for $\xi = 0$, hence, the function Π_2 has a cusp for $\xi = 0$. Around the point $z = 1$, we obtain by changing from z to $r = (z - 1)$ and by applying the binomial theorem, if necessary:

$$\overline{\Pi}_2 = u_0^2 \rho_0 p_0^{-1} [M_0' + M_1' (z-1) + (\tfrac{1}{2}) M_2' (z-1)^2 + (\tfrac{1}{3}) M_3' (z-1)^3 + \cdots]. \quad (128)$$

M_0' is determined from the boundary condition that $\overline{\Pi}_2(\xi_{*2}) = 0$.

6. The Density.

Let us put into the equation of state (30) the values:

$$p = p_0 - p_2, \qquad T = T_0 - T_1 - T_2, \qquad \rho = \rho_0 - \rho_2,$$
$$\rho_2 = \rho_0 x'^{-1} \Gamma_2, \quad (129)$$

and keep only the terms of the order $x'^{-\frac{1}{2}}$. We easily obtain:

$$\Gamma_2 \theta_1 + \Pi_2 = 0. \quad (130)$$

For the unit circle around $z = 0$ we obtain by expansion in series:

$$\Gamma_2 = N_0' + N_{2/3}' Z^{2/3} + N_1' z + N_{4/3}' z^{4/3} + N_{5/3}' z^{5/3}$$
$$+ N_2' z^2 + N_{7/3}' z^{7/3} + N_{8/3}' z^{8/3} + \cdots \quad (131)$$

One may easily check that $\Gamma_2' \neq 0$ for $\xi = 0$ (a cusp). To find the solution for the unit circle around $z = 1$, put $(z - 1) = r$, and obtain from (130):

$$C_{11}r^2\overline{\Gamma}_2 + 4\overline{\Pi}_2 = 0. \tag{132}$$

Putting:

$$\overline{\Gamma}_2 = P'_{-2}r^{-2} + P_{-1}r^{-1} + P_0' + P_1'r + P_2'r^2 + P_3'r^3 + \cdots, \tag{133}$$

we may easily find the coefficients.

Let us find the first non-vanishing term in the last boundary condition. We cannot expand our function in Taylor series around the point $\xi = z = 1$, since this point is a pole of the second order for the function $\rho_2 = \rho_2(\xi)$. The value of ρ_2 for $\xi = 1$ must be calculated directly from the equation of state. Let us expand the function around the point $\xi = 1.1$:

$$\rho_2(y'_{c2}) \sim x'^{-1} \tag{134}$$

$$\rho_2'(y'_{c2}) = (3A)^{-1}\rho_0 x'^{-1}\overline{\Gamma}_2'(\xi)|_{\xi=\xi_{c2}}. \tag{135}$$

Expanding the function $\overline{\Gamma}_2'(\xi)$ around the point $\xi = 1.1$, we easily observe that the first term $\overline{\Gamma}_2'(1.1)$ will be a constant, hence $\rho_2'(y'_{c2}) \sim x'^{-1}$, which is an acceptable order for the given approximation.

7. The Refinements of Second Approximation.

The refinements will consist of taking into account the complete tensor forms according to Prandtl's and the author's suggestions. In order to show the influence of those forms let us calculate the corrected second approximation to the pressure from the second equation of motion. Substituting into (34a) and (34b) the values:

$$u = u_0 - u_1 - u_2, \quad v = v_1 + v_2, \quad p = p_0 - p_2, \quad \rho = \rho_0 - \rho_2, \tag{136}$$

and keeping only the terms of the order x'^{-2} we obtain:

$$\partial_{y'}p_2 = \rho_0\{u_0\partial_{x'}v_1 + c^2[(\partial_{y'}u_1)^2 + 2x'(\partial_{y'}u_1\partial^2_{x'y'}u_1 - \partial^2_{y'y'}u_1\partial_{y'}v_1 - \partial_{y'}u_1\partial^2_{y'y'}v_1)]\}. \tag{137}$$

From the term $\overline{\Gamma}$ only the part of the order x'^{-1} i.e. $\partial_{y'}u_1$ was retained. Changing (137) to the variable z, putting:

$$p_{2c} = p_0 x'^{-\frac{1}{3}}\Pi_{2c}, \tag{138}$$

where the subscript "c" means "corrected," and performing all the necessary transformations gives the expression:

$$\Pi'_{2c}(z) = (\tfrac{1}{3})[10S_1'z^{7/3} + 7S_2'z^{4/3} + 4S_3'z^{\frac{1}{3}} + S_4'z^{-\frac{1}{3}}]. \tag{139}$$

As we see $p'_{2c}(\xi) \to \infty$ for $\xi = 0$, which means that the curve $p_{2c}(\xi)$ has a cusp for $\xi = 0$. The integration gives:

$$\Pi_{2c} = S_1'\xi^3 + S_2'\xi^{7/3} + S_3'\xi^2 + S_4'\xi^{\frac{1}{3}} + S_5', \tag{140}$$

where the constant S_1' must be determined from the condition that $\Pi_{2c}(\xi_{x2}) = 0$. The first non-vanishing term is of the order x'^{-2}. The form of Π_{2c} is simpler than that of Π_2.

VII. THE THIRD APPROXIMATION.

1. *The Longitudinal Velocity Component.*

Let us put into the first equation of motion the values:

$$u = u_0 - u_1 - u_2 - u_3, \quad v = v_1 + v_2, \quad p = p_0 - p_2, \quad \rho = \rho_0 - \rho_2, \tag{142}$$

$$l = ac(x'^{\frac{1}{2}} + c_1 + \mathfrak{c}_2 x'^{-\frac{1}{2}}), \tag{143}$$

where $\mathfrak{c}_2$ must be determined from test data, and keep only the terms of order $x'^{-5/2}$; we obtain:

$$\begin{aligned}
&\rho_0(u_2\partial_{x'}u_1 + u_1\partial_{x'}u_2 - u_0\partial_{x'}u_3 - v_1\partial_{y'}u_2 - v_2\partial_{y'}u_1) + u_0\rho_2\partial_{x'}u_1 \\
&\quad - 2c^2\{\rho_0\partial_{y'}u_1[(2\mathfrak{c}_2 + c_1^2)\partial^2_{y'y'}u_1 + \partial_{y'}u_2] + u_0\partial_{y'}u_1\partial_{y'}\rho_2\} \\
&\quad - \partial_{x'}p_2 - x'^{-\frac{1}{2}}c_1c^3\rho_0(\partial_{y'}u_1)^2 - x'^{\frac{1}{2}}2c_1c^2\{2\rho_0[\partial_{y'}u_1\partial^2_{x'y'}u_1 + \partial^2_{y'y'}u_1\partial_{y'}u_2 \\
&\quad + \partial_{y'}u_1\partial^2_{y'y'}u_2] + u_0(\partial_{y'}u_1\partial^2_{y'y'}\rho_2 + \partial^2_{y'y'}u_1\partial_{y'}\rho_2)\} \\
&\quad - x'c^2\{2\rho_0[\partial^2_{x'y'}u_1\partial_{y'}u_2 + \partial_{y'}u_1\partial^2_{x'y'}u_2 + \partial_{y'}u_2\partial^2_{y'y'}u_2 \\
&\quad + \partial_{y'}u_1\partial^2_{y'y'}u_3 + \partial^2_{y'y'}u_1\partial_{y'}u_3] + u_0[2(\partial^2_{x'y'}u_1\partial_{y'}\rho_2 + \partial_{y'}u_1\partial^2_{x'y'}\rho_2) \\
&\quad + (\partial^2_{y'y'}u_2\partial_{y'}\rho_2 + \partial_{y'}u_2\partial^2_{y'y'}\rho_2)] + u_1(\partial^2_{y'y'}u_1\partial_{y'}\rho_2 + \partial_{y'}u_1\partial^2_{y'y'}\rho_2) \\
&\quad - 2\partial_{y'}u_1\partial^2_{y'y'}u_1\cdot\rho_2\} + x'^{\frac{1}{2}}c^3[(\partial_{y'}u_1)^2\partial^2_{y'y'}\rho_2 + 2\partial_{y'}u_1\partial^2_{y'y'}u_1\partial_{y'}\rho_2] = 0.
\end{aligned} \tag{144}$$

It is to be noted that there are a considerable number of terms containing the variable ρ_2 in (144). Hence the influence of compressibility is a very significant one. Introducing the variable η and performing all of the necessary transformations gives:

$$\begin{aligned}
&2c^2(f'H'' + f''H') - \tfrac{1}{2}\eta H' - \tfrac{3}{2}H \\
&\quad + c^2\{2[(2\mathfrak{c}_2 + c_1^2) + c_1\eta]f'f'' + 3c_1f'^2\} + \tfrac{1}{2}(3f + \eta f')\Sigma g \\
&\quad - c^2(3f' + \eta f'' - 4c_1f'')\Sigma g' - c^2(\eta f' - 4c_1f')\Sigma g'' \\
&\quad + 2c^2\Sigma g'\Sigma g'' - Gf + [\tfrac{1}{2}(f + \eta f') - 2c^2f'f'']\Gamma_2 \\
&\quad - c^2(3f' - 2c_1f'' + \eta f'' - ff'' + 2c^2f'f'')\Gamma_2' \\
&\quad + [c^2(2c_1f' - \eta f' + ff') - c^2f'^2]\Gamma_2'' + 2c^2(\Gamma_2'\Sigma g'' + \Gamma_2''\Sigma g') \\
&\quad - \tfrac{1}{2}u_0^{-2}\rho_0p_0(3\Pi_2 + \eta\Pi_2') = 0,
\end{aligned} \tag{145}$$

$$\Sigma g = g_2 + g_3 + g_{44}, \quad \Sigma g' = g_2' + g_3' + g_{44}', \quad \Sigma g'' = g_2'' + g_3'' + g_{44}''. \tag{145a}$$

Introducing the variable η_1 gives:

$$\begin{aligned}
&f'H'' + (f'' - \tfrac{1}{2}\eta_1)H' - \tfrac{3}{2}H + [(2\mathfrak{c}_2 + c_1^2) + c_1(2c^2)^{\frac{1}{2}}\eta_1]f'f'' \\
&\quad + 3c_1c^2(2c^2)^{-\frac{1}{2}}f'^2 + \tfrac{1}{2}(3f + \eta_1f')\Sigma g - \{c^2(2c^2)^{-\frac{1}{2}}(3f' + \eta_1f'') \\
&\quad - 2c_1f''\}\Sigma g' - [c^2(2c^2)^{-\frac{1}{2}}\eta_1f' - 2c_1f']\Sigma g'' \\
&\quad + \Sigma g'\Sigma g'' - (2c^2)^{-\frac{1}{2}}Gf' + [\tfrac{1}{2}(f + \eta_1f') - f'f'']\Gamma_2 \\
&\quad - [c^2(2c^2)^{-\frac{1}{2}}(3f' + \eta_1f'') - c_1f'' - \tfrac{1}{2}ff'' + c^2(2c^2)^{-\frac{1}{2}}f'f'']\Gamma_2' \\
&\quad + [c_1f' - c^2(2c^2)^{-\frac{1}{2}}\eta_1f' + \tfrac{1}{2}ff' - \tfrac{1}{2}c^2(2c^2)^{-\frac{1}{2}}f'^2]\Gamma_2'' + \Gamma_2'\Sigma g'' \\
&\quad + \Gamma_2''\Sigma g' - \tfrac{1}{2}u_0^{-2}p_0\rho_0(3\Pi_2 + \eta_1\Pi_2') = 0.
\end{aligned} \tag{146}$$

Introducing the variable ξ gives:

$$
\begin{aligned}
&(3A_1)^{-2}f'H'' + [(3A_1)^{-2}f' - \tfrac{1}{3}\xi]H' - \tfrac{1}{3}H \\
&\quad + [(2c_2 + c_1^2) + c_1(3A)^{\frac{1}{3}}\xi](3A_1)^{-2}f'f'' + \tfrac{2}{3}c_1(2c^2)^{\frac{1}{2}}(3A_1)^{-4/3}f'^2 \\
&\quad + \tfrac{1}{3}(3f + \xi f')\Sigma g - (3A_1)^{-2}\{\tfrac{1}{3}(3A)^{\frac{1}{3}}(3f' + \xi f'') - 2c_1 f''\}\Sigma g' \\
&\quad - (3A_1)^{-2}[\tfrac{1}{3}(3A)^{\frac{1}{3}}\xi - 2c_1]f'\Sigma g'' + (3A_1)^{-2}\Sigma g'\Sigma g'' \qquad (147) \\
&\quad - (3A)^{-1}Gf' + [\tfrac{1}{3}(f + \xi f') - (3A_1)^{-2}f'f'']\Gamma_2 \\
&\quad - (3A_1)^{-2}[\tfrac{1}{3}(3A)^{\frac{1}{3}}(3f' + \xi f'') - (c_1 + \tfrac{1}{3}f)f'' + c^2(3A)^{-\frac{1}{3}}f'f'']\Gamma_2' \\
&\quad + (3A_1)^{-2}\{[c_1 - \tfrac{1}{3}(2c^2)^{\frac{1}{2}}\xi + \tfrac{1}{3}f]f' - \tfrac{1}{3}c^2(3A)^{-\frac{1}{3}}f'^2\}\Gamma_2'' \\
&\quad + (3A_1)^{-2}(\Gamma_2'\Sigma g'' + \Gamma_2''\Sigma g') - \tfrac{1}{3}u_0^{-2}p_0\rho_0(3\Pi_2 + \xi\Pi_2') = 0.
\end{aligned}
$$

Introducing the variable z gives:

$$
\begin{aligned}
&\tfrac{1}{4}A_1^{-2}[\tfrac{2}{3}zf'H'' + (\tfrac{2}{3}zf'' + f')H'] - \tfrac{2}{3}(\tfrac{1}{2}zH' + H) \\
&\quad + \tfrac{1}{4}A_1^{-2}[(2c_2 + c_1^2) + 4c_1(3A)^{\frac{1}{3}}z^{\frac{1}{3}}](3zf'' + f')f' + \tfrac{2}{3}(f + \tfrac{1}{2}zf')\Sigma g \\
&\quad - \tfrac{1}{4}A_1^{-2}[\tfrac{1}{3}(3A)^{\frac{1}{3}}z(8f' + 3z^{\frac{1}{3}}f'') - 2c_1(3zf'' + 2f')]\Sigma g' \\
&\quad - \tfrac{1}{4}A_1^{-2}[\tfrac{1}{4}(3A)^{\frac{1}{3}}z^{\frac{1}{3}} - c_1]zf'\Sigma g'' + \tfrac{1}{4}A_1^{-2}(3z\Sigma g'' + \Sigma g')\Sigma g' \\
&\quad - \tfrac{2}{3}(3A)^{-\frac{1}{3}}z^{\frac{1}{3}}f'G + \tfrac{1}{3}[f + \tfrac{2}{3}zf'(2 - A_1^{-2}f'')] - \tfrac{1}{4}A_1^{-2}f'^2]\Gamma_2 \qquad (148) \\
&\quad - \tfrac{1}{4}A_1^{-2}[\tfrac{1}{3}(3A)^{\frac{1}{3}}z^{\frac{1}{3}}(8f' + 3zf'') - (c_1 + \tfrac{1}{3}f)(3zf'' + 2f') \\
&\quad + \tfrac{2}{3}c^2(3A)^{-\frac{1}{3}}z^{\frac{1}{3}}f'(zf'' + \tfrac{1}{3}f')]\Gamma_2' + \tfrac{1}{4}A_1^{-2}\{[c_1 - \tfrac{1}{3}(2c^2)^{\frac{1}{2}}z^{\frac{1}{3}} + \tfrac{1}{3}f]zf' \\
&\quad - \tfrac{1}{3}c^2(3A)^{-\frac{1}{3}}z^{4/3}f'^2\}\Gamma_2'' + \tfrac{1}{4}A_1^{-2}[3z(\Gamma_2'\Sigma g'' + \Gamma_2''\Sigma g') + 2\Gamma_2'\Sigma g'] \\
&\quad - \tfrac{1}{3}u_0^{-2}p_0\rho_0(\Pi_2 + \tfrac{1}{3}z\Pi_2') = 0.
\end{aligned}
$$

It is a non-homogeneous linear ordinary differential equation of the second order. The homogeneous equation after all the necessary transformations takes the form:

$$z(1 - z)H'' + \tfrac{2}{3}(1 - z)H' + 2H = 0. \qquad (149)$$

It is a Gauss equation with:

$$\alpha = \tfrac{1}{6}(-1 + \sqrt{73}), \qquad \beta = \tfrac{1}{6}(-1 - \sqrt{73}), \qquad \gamma = \tfrac{2}{3}. \qquad (149a)$$

The same reasoning as in the second approximation will lead to the conclusion that both solutions of the homogeneous equation drop out. We have to find the particular solutions. Let us split the entire equation into ten non-homogeneous equations. We obtain:

$$\sum_{1}^{10}[L(H_i) + f_i(z) = 0]. \qquad (150)$$

All of these equations were solved by expansion in infinite series (see Appendix). The Hi will denote an expansion around $z = 0$, $\bar{H}i$ an expansion around $z = 1$. B.C. will denote boundary conditions. $\bar{H}i$'s were found by use of the transformation $(z - 1) = r$. All the fractional powers were expanded by the Binomial Theorem. The order of the first non-vanishing terms was found by Taylor series around $\xi = 1$ or $\xi = 1.1$ from the equations:

$$u_3 = u_0 x'^{-\frac{1}{2}} \sum_1^{10} \bar{H}_i(\xi), \tag{151a}$$

$$u_3'(y'_{*2}) = u_0(3A)^{-\frac{1}{2}} x'^{-\frac{1}{2}} \sum_1^{10} \bar{H}_i'(\xi_{*2}). \tag{151b}$$

The following remarks refer to particular functions: (H_1): the coefficient $\bar{a}_0$, which contributes to the velocity distribution along the x-axis, must be determined from the momentum considerations as was done for "a_0" in the second approximation. (H_2): $\bar{f}_0$ must be determined from momentum considerations, B.C. is fulfilled. (H_3): δ_0' from momentum considerations, B.C. is fulfilled. (H_4): $\bar{F}_0$ from momentum considerations, B.C. is fulfilled. (H_5): $\bar{B}_0$ from momentum considerations, B.C. is fulfilled. (H_6): $\bar{i}_0$ from momentum considerations, B.C. is fulfilled. H_7 drops out since the expression inside the bracket in the term containing Γ_2 is equal to zero. (H_8): $\bar{G}_0$ from momentum considerations, B.C. is fulfilled. $(\bar{H}_8)$: as was done in the expansion for Γ_2, in $\bar{H}_8$ represent $(\log x)$ by its series. The point $z = \xi = 1$ is a logarithmic singularity for the function $\bar{H}_8(z)$. The value of $\bar{H}_8$ for $z = \xi = 1$ cannot be calculated directly from the equation for $(\bar{H}_8)$. $\bar{H}_8'$ was expanded around $\xi = 1.1$. (H_9): B.C. is fulfilled. $(\bar{H}_9)$: The point $z = \xi = 1$ is a pole. $\bar{H}_9'(\xi_{*2})$ may be expanded around $\xi = 1.1$, say. (H_{10}) $\bar{N}_0$ from momentum considerations, B.C. is fulfilled.

Hence, it was possible to find all the particular integrals. In the same way higher order approximations may be obtained. They are a question of routine work. Because of that we shall discontinue the calculations at this point.

It may be noticed that the explained procedure will not be subject to a change if we preserve all the terms due to viscosity and heat conductivity or if we introduce a three dimensional flow pattern behind an infinitely long cylindrical body.

Part II will appear in the February issue.

New Fluid Which Kills Shine on Worn Clothing Developed in Laboratory.—Science, after years of research and study, has succeeded in developing an acid-and-abrasive-free liquid preparation that will remove the shine from clothing without injury to the fabric.

This new product, new in the field of dry cleaning and textiles, will be introduced to the public by the Vitatex Products Corporation, RKO Building, New York City, under the name of "Killshine."

Successfully tested and proved in several of the leading laboratories. "Killshine" reportedly not only removes shine from clothing but restores the cloth fibers to their original appearance by "waking up" the nap, and giving it lustre.

"Killshine" bears the Good Housekeeping Guaranty Seal which was granted after extensive testing.

Months of investigation in the research laboratory of the Vitatex Products Corporation were responsible for the discovery, which led to the ultimate development of "Killshine."

"Killshine" works on the same principle as cold permanent wave lotions, that is, when "Killshine" is applied to the shiny area of a piece of material, its chemical action temporarily softens the tiny fibers which normally form the nap. While these fibers are in this pliable state, they are brought back to their original position by the brushing action used in the treatment, thus restoring the nap and killing the shine. "Killshine," in effect, gives a permanent wave to the tiny fibers that form the nap on most materials and in this way removes the unsightly shine. On fabrics that have no nap, shine is usually the result of a flattening of the fibers that make up the material. "Killshine" temporarily softens these fibers and the brushing action restores them to their original shape, thus killing the shine.

One of the laboratories, the first used in proving the merits of the fluid, reported that "Killshine" has no detrimental effect on color-fast materials. The chief chemist's report is quoted as follows:

"'Killshine' will remove shine from textiles. It gives fabrics new useful life and appearance. No shrinkage of appreciable amount resulted." He further declared that "Killshine" has no irritative effect upon the skin nor a retarding influence upon the healing of an open wound. It is non-inflammable, colorless and leaves no odor.

Among the articles of clothing tested were a pair of blue serge trousers showing a decided and unsightly shine in the seat and other areas. "Killshine" was used sparingly. No hard rubbing or mechanical devices were used. After the application, and while the material was still damp, a bristle brush was used to "wake up" the nap. When the area was dry, examination showed the treatment had removed the shine and it appeared to have restored the color of the material in the trousers almost to its original shade. No injury to cloth was found nor was a "ring" discoverable even though the test fabric was somewhat soiled before the "Killshine" was applied.

Other tests were made on rayons, gabardines, worsteds and cottons. Each fabric, after treatment, was reported free of undesirable shine.

NOTES FROM THE NATIONAL BUREAU OF STANDARDS.[1]

MAGNETIC MEASUREMENT OF COMPOSITE COPPER-NICKEL COATINGS.

A convenient, nondestructive magnetic method for determining the thickness of composite copper-nickel coatings electrodeposited on steel has been developed at the Electrodeposition Laboratory of the National Bureau of Standards. The method involves the measurement of the attractive force between the plated specimen and two permanent magnets of different strengths. The values thus obtained are used, in conjunction with a set of previously determined calibration curves for each magnet, to obtain the total thickness of the coating and the relative thickness of the copper and nickel layers.

Composite coatings, in which nickel is deposited over a layer of copper, are used extensively in the automotive and other industries for the economical protection of steel against corrosion. Although the effectiveness of these coatings is usually somewhat inferior to that of pure nickel plate, the copper required is less expensive than the extra layer of nickel that would otherwise be necessary. As the protective value of the composite coatings depends largely on their thickness, it is important to have a convenient means for measuring this property. In the usual procedure, the plated article must be destroyed to provide a cross section for microscopic examination. The new magnetic method, on the other hand, eliminates the necessity of cutting the specimen and is easily and rapidly applied.

This method utilizes the principle of the Magne-gage, an instrument originally designed at the Bureau to measure the thickness of single electro-deposited coatings on the basis of the attraction between a small permanent magnet and the plated sample. The Magne-gage is essentially a spring balance, on the arm of which a magnet is suspended in contact with the coated surface. A helical spring is so attached that, when wound by means of a knob, it exerts a force tending to detach the magnet from the surface. If, as is ordinarily the case, the coating is less magnetic than the object plated, the required force, as indicated on a dial adjacent to the knob, will be greater for thinner coatings. For measurement of composite coatings, the Magne-gage is modified by the use of two magnets of different strengths. Total thicknesses of composite coatings ranging from 0.0005 to 0.003 in. can be determined within about 10 per cent, and the thickness of each component layer to an accuracy of about 15 per cent.

[1] Communicated by the Director.

WEATHER RESISTANCE OF PORCELAIN ENAMELS.

During the past few years the increased use of porcelain-enameled steel as an architectural material has accentuated the need for adequate data on the weathering resistance of various types of porcelain enamels. In 1939 the National Bureau of Standards began a long range, systematic study of the weather resistance of porcelain enamels. Specimens including both acid-resistant and non-acid-resistant enamels were supplied by sixteen cooperating manufacturers and exposed to four different climatic conditions: temperate residential, temperate industrial, temperate "salt air," and semitropical residential. Five duplicate sets of specimens were prepared. One of these was placed in storage and the other four installed at selected exposure sites in Washington D. C.; St. Louis, Mo.; Atlantic City, N. J.; and Lakeland, Fla.

The panels were installed in 1940. At the end of the first year a complete inspection was made and the results, together with preliminary work on accelerated tests, were published at that time. The second inspection of the panels at the four locations was delayed due to war conditions and was not completed until 1947 when the panels had been exposed for seven years. The more important findings may be summarized as follows: In contrast to the results of the first-year inspection, which showed most weathering effects at St. Louis and Washington, the seven-year data indicate that the conditions at Atlantic City were most severe on the acid-resistant enamels while Lakeland conditions were most severe on the non-acid-resistant compositions. The data from the St. Louis panels at seven years were not consistent with the other data because of a heavy, adherent, tar-like deposit originating from heavy smoke concentration in the test area, which was near a railway terminal. The resulting film was protective, but would not normally have been permitted to accumulate for a six-year period.

Good correlation existed between weather resistance, as measured by the average percentage of the initial gloss retained, and acid resistance, as determined by the standard Porcelain Enamel Institute test. Noticeable fading of glossy colored enamels occurred only on those panels of poor acid resistance. Where the initial coverage was complete and no mechanical damage had occurred, protection of the metal against corrosion was unimpaired after seven years for all specimens.

COMPARISON OF IONOSPHERIC DATA FROM ANTIPODAL POINTS.

For several years it has been known that ionospheric characteristics of a station in a given latitude and longitude and at a given season of the year are simulated by a station at the antipodal point for the corresponding season. If this reciprocity can be definitely established it

would effectively double the information obtained from each operating ionospheric station, and further, supply information for parts of the world where for political, economic, or geographical reasons no data are available. The pair of stations most suited for study of this principle is Watheroo, Western Australia, and Baton Rouge, Louisiana.

Comparisons on a statistical basis at the National Bureau of Standards have been made of the values observed at Baton Rouge with those calculated (1) from the Baton Rouge data, and (2) from the Watheroo data. The agreement in the two cases showed that Watheroo was nearly as efficient an indicator of Baton Rouge characteristics as Baton Rouge itself. Comparisons were also made between observed characteristics at Brisbane, on the eastern side of Australia, with the calculations from (1) the Brisbane data and (2) the Watheroo data. Study of the Baton Rouge and Brisbane results indicate that Watheroo is just as efficient for predicting Brisbane as it is for Baton Rouge.

Since Baton Rouge is as far in longitude from the antipodal point of Watheroo as Brisbane is from Watheroo itself, the discrepancies in the characteristics of Baton Rouge as calculated from Watheroo data seem to be attributable to this longitude difference alone.

EVALUATION OF HAND LUGGAGE PERFORMANCE.

A research project to develop performance tests for hand luggage has been initiated at the request of the American Association of Baggage Traffic Managers. Similar requests were later received from organizations representing bus lines and air lines. Because of excessive damage, loss, and inconvenience experienced by travelers and transportation companies from the use of luggage whose design or materials are unsuitable to withstand handling in ordinary service, means are being sought to effect improvement. The requests from common carriers were primarily for the development of a Commercial Standard for hand luggage, which would set forth minimum requirements for luggage intended for ordinary baggage service. However, little definite information on the properties and behavior of luggage was available, and it was necessary to conduct research to devise tests and provide data for determining suitable requirements.

Surveys of luggage in actual service were made in baggage rooms at passenger terminals in several cities to determine the most frequently used types of luggage, the kinds of damage occurring most often, and the weight to which travelers usually load their bags. Laboratory tests were then devised to reproduce the common kinds of damage, and a system for quantitative evaluation of damaged bags was set up. Twelve different classes of bags, purchased in the open market, were subjected to laboratory tests and to actual service on railroads. It was found that the results of the laboratory tests correlated very well with the results

of service tests, so that the performance of a given kind of bag in actual service can be predicted from the results of laboratory tests.

Four tests appear sufficient for the purposes of the Commercial Standard requested. They are a static load test; a puncture test; a repeated drop test, which uses a divided table drop-tester meeting ASTM requirements; and a handle fatigue test, using an apparatus designed and built in NBS laboratories.

A HIGHLY SENSITIVE DIFFERENTIAL MANOMETER.

A differential manometer [1] that will detect pressure differences as small as one millionth of an atmosphere over a range of total pressure from 1 mm. of mercury to 1 atmosphere has been developed by Harry Matheson and Murray Eden of the National Bureau of Standards. This instrument, which operates on the principle of the aneroid barometer, can be used to compare pressures of gases, organic vapors, and noncorrosive liquids. It is particularly useful in vapor-pressure work since less than a drop of liquid is sufficient to compare an unknown vapor pressure as a function of temperature with that of a known liquid. The rest point of the new manometer is insensitive to temperature, and accidental pressure differences as high as an atmosphere introduce only temporary hysteresis effects. The instrument is also readily portable and need not be leveled before operation.

A number of devices for measurement of absolute vacuum or pressure differences have been available to the laboratory worker. However, the majority of such gages do not operate satisfactorily in the presence of vapors. Other common disadvantages are large dead volume, inconvenience of handling or moving, difficulty or slowness of manipulation, lack of precision, and sensitivity to temperature. In connection with a study at the National Bureau of Standards of high-polymer plastics in solution, it became necessary to compare solution vapor pressures accurately. Lack of suitable apparatus for this purpose resulted in the design of the new type of differential manometer, which is expected to prove useful in other fields of research. It may also be advantageously substituted for slope gages and other commercial devices for the measurement of small differences of pressure.

The instrument developed at the Bureau measures a pressure difference as the change in the separation of a pair of nesting diaphragms indicated by an unbonded wire-resistance displacement gage. The two diaphragms are soldered together at their edges to form a very thin capsule into which is admitted one of the vapors whose pressure is to be compared. The other vapor surrounds the capsule externally.

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[1] For complete technical details see "A highly sensitive differential manometer," by Harry Matheson and Murray Eden, *Rev. Sci. Insts.*, August, 1948.

One of the diaphragms is attached to a stationary support through which vapor enters the capsule; the other is connected to the displacement gage. The four resistance wires of the gage are arranged to form a Wheatstone bridge, in which an increased separation of the diaphragms elongates one pair of wires and shortens the other pair. A decrease in the diaphragm spacing has the opposite effect. The resistive unbalance as indicated by a galvanometer gives an accurate measure of the separation of the diaphragms, and thus of the pressure difference. The capsule and gage are sealed in a vacuum-tight chamber.

Tests of the new manometer at the National Bureau of Standards have shown that its sensitivity is not appreciably altered when the active elements are used at temperatures between 20° and 50° C., for values of absolute pressure between 1 mm. of mercury and 1 atmosphere. The elements can withstand a differential pressure of 1 atmosphere where the greater pressure is external to the capsule, or a pressure difference of 0.2 atmosphere in the other direction. The equality of two pressures may be found to 0.001 mm. of mercury; the direction and magnitude of a pressure inequality may be determined between 0 and ± 0.5 mm. of mercury. The indicated rest point remains constant within 0.005 mm. of mercury for periods of at least one day, even though the device may be subjected during this time to pressure differentials of the order of 0.2 atmosphere and temperature variations of several degrees.

Thermometer Listens to Temperature (*Science Illustrated*, Vol. 3, No. 12).—A mysterious noise that once baffled radio and television engineers has led to the design of the world's most accurate thermometer.

At the University of Chicago's Institute of Metals, two young physicists, Drs. John B. Garrison and Andrew W. Dawson, have put the noise of bouncing atomic particles to work. Heat, they reasoned, is simply the motion of molecules. As the molecules move, they hit free electrons, which are thousands of times lighter and therefore bounce off like billiard balls. As the electrons bounce, they set up small electrical impulses, which can be amplified and measured. Such amplified impulses can be heard as noise. But they can also be put to use telling temperature, since they increase with the amount of molecular motion, and therefore, with the amount of heat. The complicated instrument is called the "noise" thermometer. It converts the noise into pulses which are visible on an oscilloscope, and from which accurate temperatures can be calculated.

It will give accurate readings between a fraction of a degree above absolute zero (abs. zero is minus 459.6 F., the point where molecular motion stops, and the noise vanishes) to 5,000 F. At the high end of the range the thermometer is more accurate than any other type. It is completely unaffected by pressure and other phenomena.

Too cumbersome and expensive to be used for measuring the temperature of common things like baby's bath, the thermometer finally provides an answer to the high temperature worries of jet-engine engineers, who have had great trouble determining the temperature of the exhaust gases and the internal parts of the engine. Other uses will be in the steel and glass industries.

Flood Forecasting Made Easy (*Science Illustrated*, Vol. 3, No. 12).—People who live in river valleys soon will get quicker warnings of threatening floods. The U. S. Weather Bureau will give the faster warning service by making nation-wide use of its latest invention. Called the Electronic Flood Routing Machine, the device quickly and automatically computes the down-stream effect of heavy inflow into any river or headwaters stream.

The machine's three inventors, R. K. Linsley, M. A. Kohler, and L. W. Foskett, utilized a long-recognized principle: The flow of electric current and the flow of water are analogous. Fundamentally, this means an electric circuit—its voltage, amperage, and resistance of its conductors—can impersonate the characteristics of flowing water. Machines that can produce this facsimile effect are called electric analog computers.

Here's how it's done in the case of the flood routing analog machine. Data on the inflow of the river (water pouring into it from tributaries or from rainfall) are supplied to the machine's input unit in the form of a graph. The graph in turn guides the operator in regulating an electric current. The electricity masquerades as the water flowing down stream, aping all of its characteristics. This electric current instantaneously produces another graph in the output unit. The latter graph gives an accurate preview of the river's height when it reaches down-stream. At present such a flood forecast is manually computed.

THE FRANKLIN INSTITUTE.

STATED MONTHLY MEETING, WEDNESDAY, DECEMBER 15, 1948.

The Stated Monthly Meeting of The Franklin Institute was held on December 15, 1948 in the Lecture Hall of the Institute. The President, Mr. Richard T. Nalle, called the meeting to order at 8:15 P.M. There were approximately 200 persons in attendance.

The President stated that the minutes of the monthly meeting for November had been printed in full in the December issue of the JOURNAL OF THE FRANKLIN INSTITUTE, and if there were no corrections or additions, the minutes would stand approved as printed. There was no dissent.

Mr. Nalle then called on Dr. Henry B. Allen, Executive Vice President and Secretary, who presented the following names as nominations for election to the Board of Managers to serve during the years 1949, 1950, and 1951, in accordance with the By-Laws, Article IV, Section 5:

RUPEN EKSERGIAN
W. H. FULWEILER
HORACE P. LIVERSIDGE
MALCOLM LLOYD, JR.
MARSHALL S. MORGAN
EDWARD WARWICK
E. G. BUDD, JR.
WILLIAM J. MEINEL

The nominations were then closed.

Dr. Allen then announced that our Christmas Week Lectures, which are given for young people through the James Mapes Dodge Foundation, would be held on December 28th and 29th at 4:00 P.M. Dr. Ira M. Freeman of Rutgers University will present these lectures which will be entitled "Invitation to Experiment."

The Secretary reminded the members present of the very interesting exhibit and demonstration from the Brookhaven National Laboratory, now being held in the Museum, and which will remain through the month of January, 1949.

The President then introduced the speaker of the evening, Mr. Jacob Rabinow, Chief, Mechanical Engineering Section, National Bureau of Standards, Washington, the subject of his talk being the "Magnetic Fluid Clutch." Mr. Rabinow related the story of the development of this device, discussed its technical characteristics, and pointed out some of the entirely new uses to which the magnetic fluid can be applied. The lecture was illustrated with demonstrations.

The President adjourned the meeting at 9:45 P.M. with a rising vote of thanks to Mr. Rabinow.

HENRY B. ALLEN,
Secretary.

COMMITTEE ON SCIENCE AND THE ARTS.

(Abstract of Proceedings of Stated Meeting held Wednesday, December 8, 1948.)

HALL OF THE COMMITTEE,
PHILADELPHIA, DECEMBER 8, 1948.

DR. JULIAN W. HILL, in the Chair.

The following reports were presented for final action:

No. 3168: "ENIAC."

This report recommended the award of a Howard N. Potts Medal each to John William Mauchly and J. Presper Eckert, Jr., both of Philadelphia, Pennsylvania, "In recognition of their design and construction of the ENIAC, the first large-scale, general purpose, digital electronic computer, a machine of great accuracy which makes feasible the application of mathematics to problems which hitherto had either to be ignored or else solved in a much more laborious way."

No. 3155: Tank Gun Stabilizer.

This report recommended the award of a Howard N. Potts Medal to Clinton Richards Hanna, of Pittsburgh, Pennsylvania, "In recognition of his initiative in the conception and development of the Tank Gun Stabilizer, the value of this device in attaining accuracy of fire while the tank is in motion over rough terrain and in securing a greater number of aimed shots than was formerly possible."

JOHN FRAZER,
Secretary to Committee.

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MUSEUM NOTES.

UNIQUE MODEL RAILROAD.

The model railroad appeals to young and old. Because of the widespread interest in this instructive hobby many thousands of visitors are expected at the Museum to view the outstanding model of its kind which is being lent by the Baltimore and Ohio Railroad.

This model railroad will fascinate everyone who takes delight in watching a miniature railroad in operation. On a platform 16 by 46 feet is reproduced the three-track system similar in operation to the B & O's Cumberland Division, which is marked by high density traffic. Through employment of proper signal control and the use of cross-overs it is possible to perform on three tracks those operations which usually require four.

When a passenger train overtakes a freight, moving in the same direction and both running on the same outside track, the freight train is switched to the inside track and keeps on going. When the passenger train has passed, the freight is switched back to the outside track behind the passenger, so that neither train loses any time. It will be seen that the center track serves as a continuous passing siding which permits the freight to keep on the move, a decided improvement on the two-track system with its standard passing sidings where freight must suffer costly delays while waiting for passenger trains to pass.

This model is unique in employing the same method as the B & O Railroad does. The center track is wired for signal operation in both directions. There is positive train control, so that it is impossible for a train to pass a "block" unless the block ahead is clear. Nor can any train proceed if a switch is set against it. Spectators will find much to interest them in watching the changing aspects as the trains approach and pass the signal bridges.

The trains shown in operation are "O" gauge patterned after standard B & O trains (that is, they are scaled to ¼ in. to the foot). The locomotives comprise the new Diesel powered streamliners, heavy Pacific,—titans of the fast freights—Mikados and Santa Fe's, while almost every type of coach and car is reproduced.

This remarkable miniature does not show the mainline operation only. There is a busy yard and engine terminal. Model railroad enthusiasts will devote most of their attention to the junction of the main-line with the yard. Here fast streamliners and freights freely intermingle in a complicated web of traffic.

Typical buildings are dotted along the "right of way" and at terminals. Stations, section houses, coal tipple, turntable, round-house, sand bins, water tanks, and signal towers are faithful reproductions of actual buildings. Down to the smallest detail, this miniature railroad is a reproduction of the real thing.

Let us issue a word of caution against those who are accustomed to the dizzy speed at which toy trains rattle around a Christmas tree. Do not be deceived by the speed of the trains on this miniature railroad. Only a small boy will demand that the trains be speeded up. Faithful to railroad practice, these little trains are running at scale speed.

As we mentioned, the scale is ¼ in. to the foot. Consequently this is one-forty-eighth of actual size. The three-track loop is nearly a mile around, that is to say, it is one-forty-eighth of a real mile of real railroad. If the spectator will take the trouble to time the trains and finds that one completes the lap in 44 seconds, then the train is actually running a mile in 45 seconds, or 75 miles an hour—which is fast time for real trains. If the train completes the lap in 58 seconds, then it is doing 60 miles an hour. Just remember that these little trains, one-forty-eighth the size of their prototypes, are travelling in a landscape built upon a similar scale. Therefore, if they travel a scale mile in 60 seconds, they are actually doing 60 miles an hour. This is known as "scale speed" in the fraternity of model railroaders.

This fascinating miniature railroad will be opened to the public February 11th and will remain on exhibition until April 3rd, 1949.

LIBRARY NOTES.

The Committee on Library desires to add to the collections any technical works that members would wish to contribute. Contributions will be gratefully acknowledged and placed in the library. Duplicates received will be transferred to other libraries as gifts of the donor.

Photostat Service. Photostat prints of any material in the collections can be supplied on request. The average cost for a print 9 × 14 inches is thirty-five cents.

The Library and reading room are open on Mondays, Tuesdays, Fridays, and Saturdays from 9 A.M. until 5 P.M., and Wednesdays and Thursdays from 2 P.M. until 10 P.M.

RECENT ADDITIONS.

AERONAUTICS.

AERONAUTICAL Engineering Index, 1947. 1948.
CONWAY, H. D. Aircraft Strength of Materials. 1947.

ARCHITECTURE AND BUILDING.

EMERICK, ROBERT HENDERSON. Basic Refrigeration and Air Conditioning. 1948.

BIOGRAPHIES.

SHIPTON, CLIFFORD K. Isaih Thomas. 1948.

CIVIL ENGINEERING.

THOMAS, W. NORMAN. Surveying. Fourth Edition. 1948.

FUEL.

SHNIDMAN, LOUIS. Gaseous Fuels. 1948.

GEOLOGY.

LONGWELL, CHESTER R.; KNOPF, ADOLPH AND FLINT, R. F. Physical Geology. Third Edition. 1948.
WORCESTER, PHILIP G. A Textbook of Geomorphology. Second Edition. 1948.

MANUFACTURE.

VANDERBILT, R. T., COMPANY. Vanderbilt Rubber Handbook. Ninth Edition. 1948.

MATHEMATICS.

CRANK, J. The Differential Analyser. 1947.
GOODSTEIN, R. L. A Textbook of Mathematical Analysis. 1948.
HARDY, GODFREY H. AND WRIGHT, E. M. An Introduction to the Theory of Numbers. Second Edition. 1945.
HARVARD UNIVERSITY. COMPUTATION LABORATORY. Proceedings of a Symposium on Large-scale Digital Calculating Machinery. 1948.
KELLEY, TRUMAN LEE. The Kelley Statistical Tables. Revised Edition. 1948.

MECHANICAL ENGINEERING.

BROWN, GORDON S. AND CAMPBELL, D. P. Principles of Servomechanisms. 1948.
COLVIN, FRED HERBERT AND STANLEY, FRANK A. Turning and Boring Practice. Third Edition. 1948.
FRANKE, RUDOLF. Vom Aufbau der Getriebe. Band 1. 1943.
GRUNDY, REGINALD HERBERT. The Theory and Practice of Heat Engines. 1947.
HELDT, P. M. High Speed Combustion Engines. 1948.
TAYLOR, CHARLES FAYETTE AND TAYLOR, E. S. Internal Combustion Engine. 1948.

PETROLEUM.

SACHANEN, ALEXANDER NICHOLAS. Conversion of Petroleum. Second Edition. 1948.

PHYSICS.

DOOLITTLE, JESSE SEYMOUR AND ZERBAN, ALEXANDER H. Engineering Thermodynamics. 1948.
FEDERHOFER, KARL. Graphische Kinematik und Kinetostatik. 1932.
TOWNSEND, JOHN S. E. Electrons in Gases. 1947.

RADIO.

GOLDMAN, STANFORD. Frequency Analysis, Modulation and Noise. 1948.
HAMILTON, D. R.; KNIPP, J. K. AND KUPER, J. B. H. Klystrons and Microwave Triodes. 1948.
VALLEY, GEORGE E. AND WALLMAN, HENRY. Vacuum Tube Amplifiers. 1948.

SANITARY ENGINEERING.

BAKER, MOSES NELSON. The Quest for Pure Water. 1948.

SCIENCE.

COHEN, I. BERNARD. Science, Servant of Man. 1948.
STRUIK, DIRK J. Yankee Science in the Making. 1948.

PUBLICATIONS RECEIVED.

Einfuhrungsmathematik für Physiker, by Wilhelm Lenz. 95 pages, 21 × 15 cm. Hannover, Germany, Wolfenbutteler Varlagsanstalt, 1947. No price (Paper).

Cybernetics—or Control and Communication in the Animal and the Machine, by Norbert Wiener. 194 pages, 16 × 24 cm. New York, John Wiley & Sons, Inc.; Paris, Hermann et Cie; 1948. Price, $3.00.

Radio Engineering, by E. K. Sandeman. 775 pages, 14 × 23 cm., plates, tables and drawings. New York, John Wiley & Sons, Inc., 1948. Price, $6.50.

Education for Professional Responsibility, A Report of the Proceedings of the Inter-Profes-

sions Conference on Education for Professional Responsibility held at Buck Hill Falls, Pennsylvania, April 12, 13 and 14, 1948. 207 pages, 14 × 21 cm. Carnegie Press, Pittsburgh, 1948. No Price.

Principles of Physics III, by Francis Weston Sears. 369 pages, 15 × 23 cm., drawings and illustrations. Massachusetts, Addison-Wesley Press, Inc., 1948. Price, $4.50.

Cobalt, by Roland S. Young. 181 pages, 15 × 24 cm., tables and illustrations. New York, Reinhold Publishing Corp., 1948. Price, $5.00.

Theory and Practice of Heat Engines, by Virgil Moring Faires. 388 pages, 16 × 24 cm., tables, drawings and illustrations. New York, Macmillan Co., 1948. Price, $5.00.

Annual Reports of the Progress of Chemistry for 1947, issued by The Chemical Society. 327 pages, 14 × 21 cm., London, The Chemical Society, 1948. Price, $5.00.

Photographic Emulsion Technique, by T. Thorne Baker. Second Ed., 336 pages, 14 × 20 cm., tables, drawings and illustrations. Boston, Mass., American Photographical Publishing Co., 1948. Price, $7.50.

The Mathematical Solution of Engineering Problems, by Mario G. Salvadori. 245 pages, 16 × 23 cm., tables. New York, McGraw-Hill Book Co., Inc., 1948. Price, $3.50.

Fourier Technique in X-Ray Organic Structure Analysis, by A. D. Booth. 103 pages, 14 × 22 cm., drawings and illustrations. Cambridge University Press; New York, The Macmillan Co.; 1948. Price, $2.75.

Electromagnetic Waves and Light, First Part, by Charles F. Meyer. 83 pages, 14 × 22 cm., drawings, tables and illustrations. Michigan, Edwards Brothers, Inc., 1948. No price (Paper).

The Solar System Analysed—Old Laws from a New Angle, by F. C. Attwood. 88 pages, 13 × 19 cm., drawings. New Zealand, The Dawson Printing Co., Ltd., 1947. No price.

BOOK REVIEWS.

AN INTRODUCTION TO COLOR, by Ralph M. Evans. 340 pages, 19 × 25 cm., plates and drawings. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd.; 1948. Price, $6.00.

The book herein reviewed is written in a manner to please the many people who have different requirements as to their knowledge and use of color. Writing in words having common speech usages, the author presents the fundamentals of the science and art of color, with the assumption of no mathematical maturity on the part of the reader beyond high school mathematics. Completely descriptive and non-mathematical, the many complex facts associated with color are presented through the medium of graphs, color photographs, black and white cuts, and of course a well written and easily understood text.

Those who work with color know that this subject has three main avenues of approach: physics, psycho-physics, and psychology. Restated in another form, the subject can be thought of as being a study of the effects of the properties of colored lights, the properties of vision, and the action of the mind in interpreting color. The three phases are interrelated, and a comprehensive book on color, such as this work is, must present the subject as roughly divided into the three sections noted above. The sources of light and the objects and substances that modify it are considered as pertaining to the physics of color, or the quality of the light as such without reference to any observer. The psychophysical aspects of color deal with the evaluation of purely physical light by mechanical means in terms of an average observer, which is merely a concept developed as a result of a large number of measurements on a large number of people. The psychology of color considers the relationship between color as calculated for a standard observer and color as perceived in the mind. This relationship is important, especially since the same light is not always seen as the same color.

The chapters on the physical nature of light start from "scratch," and introduce the concept of radiant energy and spectral distribution, describing the various common light sources with their spectral distributions. The elements of illumination, reflection, refraction, lenses and non-selective transmission are considered in detail prior to the introduction of the

problems of selective transmission and reflection for the cases of glass, filters, metal surfaces, and multi-layers.

Color vision is introduced through a study of the physiology of the eye and a discussion of reaction of the eye to different colors. The standard systems of color description are outlined: for the case of the so-called mental variables, hue, saturation and brightness, as worked out by the Munsell System; for the more accurate description, in terms of the ICI primaries and the ICI color triangle; and the less commonly used Ostwald system. Applications to illumination, color mixing, paints and pigments, photography, painting, design and abstraction are considered.

The book could readily be subtitled "The Magic of Light and Colors." It should be useful to architects, lighting consultants, interior decorators, stage directors, stage set designers, stage lighting consultants, and even the housewife who is interested in a better and fuller living through the medium of color.

S. CHARP.

MICROWAVE RECEIVERS, edited by S. N. Van Voorhis. 618 pages, 16 × 23 cm., tables, drawings and illustrations. New York, McGraw-Hill Book Co., 1948. Price, $8.00.

The book "Microwave Receivers" presents in twenty-one chapters written by nineteen different authors and edited by one of them, S. N. Van Voorhis, a summary of the theory and principles employed by the M.I.T. Radiation Laboratory staff in the design, construction, and testing of radar microwave receivers. The book deals primarily with four basic receiver types: the superhetrodyne, the superregenerative receiver, the tuned radio frequency receiver, and the crystal video receiver, each of which has its particular field of application.

As pointed out in the introduction, the major problems with which the radar microwave receiver designer is concerned are the problem of attaining the highest possible signal-to-noise ratio and the problem of designing high-gain wide-band amplifying systems capable of handling signals of an extremely wide dynamic range. In addition, the designer of equipment for military applications must be guided in his selection of circuits by the principle that field adjustments and maintenance must be kept as simple as possible.

The treatment of the noise problem and of considerations affecting signal-to-noise ratio pervades the entire volume, and probably represents the most complete presentation of such information in any receiver text available. From this angle alone the book is worth the attention of the receiver designer.

In its application, the material presented is not limited to the design of radar equipment. It will be of interest to anyone faced with the many design problems encountered in wide-band high-gain systems.

Practically all of the theory introduced in the text is demonstrated in practical applications in the several chapters devoted to the discussion of receivers designed, constructed and tested at the Radiation Laboratory. These chapters devote much space to the practical side of the theory application, both from an electrical and a mechanical viewpoint.

The book contains numerous bibliographical references in the footnotes. These should be of considerable help to the serious reader.

ROBERT S. GRUBMEYER.

PRINCIPLES OF SERVOMECHANISMS, by Gordon S. Brown and Donald P. Campbell. 400 pages, 15 × 24 cm., tables, and drawings. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd.; 1948. Price, $5.00.

In this new textbook by Professors Brown and Campbell, the authors bring together in a most authoritatively written manner, the best of the theoretical and design principles developed over the past decade in the field of closed loop control systems. While intended primarily for the use of both senior and graduate students in electrical, mechanical, aeronautical and chemical engineering, the book should be of interest to the scientist, the engineering teacher, and the practicing engineer, especially if the latter is at all concerned with the design or analysis of servomechanisms and control devices.

The subject of closed loop control systems, and in particular servomechanisms, has seen tremendous growth in the past few years, especially as a result of the past war. In this same period there were proposed numerous unified methods for the successful design of these devices. The early work was based on analysis which only served to help one understand a system. Then the emphasis went on to cut-and-try procedures when it became apparent that not enough was known of design methods. But neither of these methods considered the dynamics of the systems, and in fact, neither of these methods could serve to open new doors to help enlarge the scope of knowledge of the design of servomechanisms. Now in this book there is presented a new concept—a new approach to the old problems—which the authors have appropriately termed "guided conception or dynamic synthesis." This is the central theme of the book.

The synthesis considers two avenues of approach: the transient response and the sinusoidal response, each furnishing useful data, each having certain advantages and disadvantages. Of the two methods, it is shown that the sinusoidal response, while somewhat more difficult of visualization, tells the more complete story in the dynamic synthesis of servomechanisms.

The dynamics of elementary control systems are introduced first on an elementary level, then the theory is extended to higher levels of analysis and synthesis and a consideration of successively more complex control problems. Features of Laplace transforms applicable to the servo loop problem are introduced as required, this method serving to help define the system transfer function. Frequency response characteristics are considered in detail, special emphasis being placed upon techniques for studying the gain characteristics independent of the frequency functions. These studies are carried out both for synthesis using linear coordinates and for synthesis using logarithmic coordinates. The latter system of synthesis is shown to be advantageous in most cases, although it is strongly pointed out that both methods are merely two tools of synthesis, each having its own useful sphere in the servo design problem.

The designer often is faced with the problem of approximating or specifying the transient response from the frequency response, not at all a simple problem even for a simple system. A most valuable chapter, therefore, is devoted to an approximate method for performing the direct and inverse transformations.

A large collection of theoretical and design problems is included, as well as a bibliography of the most important papers published up to the end of 1947.

S. CHARP.

PHOTOELASTICITY, by Max Mark Frocht. Volume II, 505 pages, 16 × 23 cm., tables, drawings and illustrations. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd.; 1948. Price, $10.00.

Dr. Frocht's *Photoelasticity, Volume II*, is a continuation and extension of his previous publication, *Photoelasticity, Volume I*, published in 1941. The second volume deals with more advanced theories, advanced techniques and the latest developments in the field of photoelasticity.

The material in Volume II covers three phases of photoelasticity. First, there is a brief introduction to the theory of elasticity for engineers in general and photoelasticians in particular; second, a treatment of advanced methods of two-dimensional photoelasticity; and third, a treatment of the methods and techniques of three-dimensional photoelasticity.

The brief introduction to elasticity presents some of the concepts of elasticity with a clarity which is not usually found in the mathematical treatises on elasticity. Dr. Frocht's treatment of the compatibility equation and the elementry stress functions is noteworthy in this respect. Some problems discussed in the section on elasticity include radial stresses in the semi-infinite plate, radial stresses in wedges and beams and stresses in circular disks. A parallel study of each problem is made both mathematically and photoelastically, and the results are presented in the form of comparative stress patterns.

In Volume I, Dr. Frocht described several ways of determining the sum of the principal stresses by methods dependent upon photoelastic data. In Volume II, experimental methods are outlined for determining the sum of the principal stresses and for the determination of isopachic patterns. Several types of lateral extensometers are described for this purpose.

In connection with the methods for finding principal stresses, the section dealing with the numerical solution of Laplace's equation gives a powerful yet practical method for determining the sum of the principal stresses within a given region of a two-dimensionally stressed body provided that the boundary stresses are known. Dr. Frocht's treatment of the numerical solution of Laplace's equation supplemented by a section on conformal transformations offers a useful tool for the solution of photoelastic problems and mathematically similar problems which arise in heat transfer, elasticity, hydrodynamics and electricity.

Dr. Frocht treats the subject of three-dimensional photoelasticity from the standpoint of theory, techniques, and application. The stress optic law in three dimensions is discussed and demonstrated by direct photoelastic evidence. Laboratory techniques are described for preparing the three-dimensional models, freezing or fixing the elastic pattern in the model and slicing the model without disturbing the frozen pattern. An appendix discusses a new material, Fosterite, which is particularly suitable for three-dimensional photoelastic stress analysis employing frozen stress patterns.

The theory of oblique incidence is discussed, and its use is demonstrated by photoelastic determination of the shearing stresses on the boundaries of shafts subjected to pure torsion. It is also demonstrated that the method of oblique incidence may be used to separate principal stresses in two-dimensional problems. The scattered light technique is also described, and it is shown that the technique may be used to determine the shearing stresses in members subjected to pure torsion.

The clearness with which Dr. Frocht has presented the material in Volume II is matched only by the excellence of the typography. Numerous illustrations and the emphasis of important passages by varying the style of type facilitates reading or comprehensive study.

The photoelastician or any scientifically trained person will find that Dr. Frocht's fine presentation of such timely and useful information will be a valuable addition to his library.

R. H. MARVIN.

PUBLIC HEALTH ENGINEERING, by Earle B. Phelps and others. 655 pages, 14 × 22 cm., tables, drawings, and illustrations. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd.; 1948. Price, $7.50.

This new and expanded edition of an older work by Professor Phelps is aptly described by its subtitle "A Textbook of the Principles of Environmental Sanitation." As a textbook it can be used primarily in survey courses in this rather new phase of what had been called sanitary engineering. In addition to its use in certain college and university courses, this book should have strong appeal for the public-minded engineer who has already learned *how* to design and build and who through the medium of this book, can learn in the light of present day knowledge of sanitary science, *what* to build and why. This emphasis is not so much on the engineering of particular projects, as the title may infer, but rather on the public health engineering aspects of engineering construction and design.

Public health engineering can be technically defined as the application of preventative medicine, through government functions, for social ends, and through the application of basic engineering principles and concepts. In a subject so important to the public welfare, it is inevitable that engineering should play a most important role. Public health, as is shown in this book, deals largely with the problems of man's relationship to his environment and the effects of the environment on man's development and well being. It has as its task the problem of assuring a maximum of favorable contacts and a minimum of unfavorable contacts.

Written primarily as a survey of one large part of the field, the subject matter of the book is approached through chemistry and the biological sciences, especially bacteriology and physiology. These sciences are not discussed in the technical terms of the workers in these fields, but rather in descriptive terms as is befitting and necessary for a survey textbook which must serve the wide and varied backgrounds of its readers, who may be engineers, medically trained students of public health, and public health officers of municipalities.

Two major subdivisions are presented, namely, air contacts and water contacts. The former category is concerned with man's relation to the atmosphere, in its most general terms. Consideration is given to the fundamental problems of: protective housing; the thermal envi-

ronment (heating and air conditioning); the luminous environment (light and energy transmission and utilization); noise and smell, both local and municipal; and airborne contacts with certain diseases which may be established with insect carriers such as mosquitoes and with mechanical carriers such as flies.

Public health as it is affected by man's water contacts is somewhat a different problem from that of the previous theme. Here there is less intimate and direct contact, but whereas air contact is a short-time one (bad air can be changed), water contact is a long-time and long-distance contact. This statement can be explained by the example of a municipal water supply which is obtained from a geographically distant source far removed from its use point and which has been stored in reservoirs or water towers for a long period prior to use. The water contacts of man are seen to be those concerned primarily with water supply, water drainage and sewage disposal. The water stream is considered as a whole unit, with its sources and seasons and geographical distribution. Of prime importance are the problems faced daily by all public health officers, namely those of the uses and abuses of water streams, water quality, testing and standards of quality with reference to end use, and the treatment of polluted water as employed in water purification and sewage treatment. These subjects are all treated in a descriptive manner, but fully adequate for the purposes of the book.

The author notes in his introduction that this book was originally evolved as the result of a lecture course on this subject. Unfortunately the style of writing, especially in Part I of the book, is a carry-over from the lecture platform, and while there is clarity of expression, the reading is somewhat jerky. This slight criticism of the form of expression does not, of course, detract from the technical worth of the book.

S. CHARP.

NATURAL PHILOSOPHY THROUGH THE EIGHTEENTH CENTURY AND ALLIED TOPICS. (Commemoration Number to mark the 150th Anniversary of the foundation of *The Philosophical Magazine*), edited by Allan Ferguson. 164 pages, 17 × 25 cm., Port. London, Edward Arnold & Co., 1948. Price, $3.00.

In our day, when scientific workers complain that the volume of periodical literature is so extensive that it is no longer possible to keep pace with it, we find it difficult to realize that there was a time when American scientists complained at the lack of technical literature. Joseph Priestley lamented his deprivation of periodical sources of information as much as he regretted his isolation from co-workers. At that time, one of those periodicals whose belated appearance in this country was eagerly awaited was the *Philosophical Magazine*.

It is with some surprise that one realizes this familiar British journal has reached the ripe old age of 150 years. Time has not overtaken and passed this scientific journal, for in the passage of the years it has retained a spirit and vigor that has enabled it to march in step with time. When the reader recalls that the *Philosophical Magazine* has been entirely independent and has never been the organ of any university, society, or institution but has obtained its international reputation entirely upon the efforts of its publishers and the abilities of the writers who chose its pages for the publication of their work, our admiration is mingled with awe.

Beginning with the praiseworthy intention of presenting "as early an account as possible of everything new or curious in the Scientific World," the *Magazine* has far exceeded this modest claim. As the nineteenth century advanced it became more and more a journal of research, and as such it offered its readers some remarkable papers. In its pages appeared Hamilton's series of papers on the extensions of the notion of quarternions; the paper of Michaelson and Morley on the relative motion of matter and ether; no less than eighty-one of J. J. Thomson's papers; many of Kelvin's; and Rutherford's papers on the artificial transmutation of the elements. Bohr, Moseley, Chadwick, and Millikan were among its contributors.

But these are the "stars." There has been so much else that was honest, interesting, and accurate that we can only hope the work begun in 1798 by Richard Taylor will continue for the instruction of still further generations of scientific workers. It is fitting that the commemorative issue should provide an adequate historical summary of the achievements in natural philosophy during the century which saw the birth of this perennial journal.

THOMAS COULSON.

FUNDAMENTALS OF ELECTRIC WAVES, by Hugh Hildreth Skilling. Second edition, 245 pages, 15 × 24 cm., drawings. New York, John Wiley & Sons, Inc., London, Chapman & Hall, Ltd.; 1948. Price, $4.00.

For those who are unfamiliar with the first edition, this book may be described as an elementary text on electromagnetic theory illustrating its application to related items such as antennas and waveguides. It serves as an excellent introduction to the theories necessary to the treatment of microwave propagation problems. Its greatest use will be found among undergraduates in communications engineering and physics. Although it is unlikely to become a reference for practicing engineers, the book may be useful in refresher courses.

The author has used vector notation throughout the book and takes care to establish an understanding of vector operations in the beginning pages. These mathematical principles are applied to classical electrostatic field problems and used to illustrate the familiar laws of electricity. One chapter is devoted to the interpretation of magnetic field theory. A great many examples have been given to aid in visualizing these principles. Experimental evidence has been collected and tabulated to make more plausible the explanation of important ideas such as Maxwell's Hypothesis.

The subject of waves is introduced approximately half way through the book and is treated from the point of view of reflection, boundary conditions and skin effect. The concept of radiation is applied to antennas of the conventional type, that is, medium wavelength.

In the last quarter of the book waveguides are introduced, and mathematical principles presented earlier are used to analyze the modes of propagation. The last chapter, new with the second edition, gives the modifications of Maxwell's equations necessary for the treatment of an ionized region such as the ionosphere. The case of oblique reflection is taken as an example.

Two useful tables are provided on the insides of the front and back covers listing units and symbols as well as formulas and theorems of vector analysis applied to electrical phenomena. These tables have been reorganized and improved over those of the first edition.

The new edition represents a timely revision. In the interest of clarity many of the plates have been changed to better illustrate the text. The printing of vector notation has been improved, and the vector diagrams have been made stronger and clearer. Numerous new problems based on the text have been added. The listing of references in the footnotes is more complete. The short chapter on dielectric materials has been omitted.

During six years of use the original text was condensed and new material added. This revision has actually increased the worth of the second edition more than its additional sixty pages would indicate.

C. W. HARGENS.

NUCLEAR RADIATION PHYSICS, by R. E. Lapp and H. L. Andrews. 487 pages, 15 × 22 cm., tables, drawings and illustrations. New York, Prentice-Hall, Inc., 1948. Price, $6.00.

The authors of this new book on nuclear physcis present some of the most recent problems of nuclear radiation from the modern viewpoint. This book, written originally as an elementary manual in radiologic safety, has been developed and expanded into a text book in nuclear physics, with essentially a non-mathematical approach, suitable for use in a survey course in this subject on an undergraduate level, or perhaps as a text book in radiologic and nuclear physics for physicians. The phenomena of nuclear physics, as authoritatively presented here by the authors, is also of interest to those working in the biological sciences and to the users of radio isotopes in industry.

Early in the book the experimental evidences of the particle theory and the wave theory are presented, and their duality through the concept of wave mechanics helps to explain some of the facts previously accepted on an empirical basis. Because the physics of the nuclear particles is, to a certain extent, based upon classical concepts, consideration is given to the kinetic theory of gases, and to the motion of particles and the forces on them in electric and magnetic fields.

Nuclear radiations, produced naturally and artificially, are considered in detail, with emphasis on their activity and production according to the most modern devices. The α, β,

and γ particles are described, all with a minimum of mathematics, but with complete clarity as to their fundamental reactions. These are determined and measured by a wide variety of experimental techniques, utilizing ionization chamber instruments, Geiger-Mueller counters, photographic means, and others.

A modern, consistent book on radioactivity and nuclear structure must of necessity spend some time on the nuclear reactions incidental to the currently important phenomena of nuclear chain reactions and nuclear reactors or piles, and their control through the neutron. While not in great detail, the atomic power pile, as exemplified by the thermal reactor, is examined in the light of the possible development of a practical power plant. No conclusions are reached as to the design of such reactors, but some illustrative examples show that fantastically large quantities of raw materials, in terms of our presently available supply, would be necessary to operate a large-scale power plant.

Discussions of the physics of nuclear particles would be incomplete without consideration to the human value and to the use of radioactive materials as tracers. The principles of this powerful research method are outlined and each of the necessary steps is described with its present serious difficulties. The use of tracers in the biological sciences is described but reference is also made to many other applications of this new and growing technique.

At the end of each chapter is a bibliography of reference books and articles. This list, of especial interest to the more serious student, in addition to the few problems, some of which require considerable ingenuity for a complete solution, should keep the advanced student actively interested in the research phases of nuclear physics.

S. CHARP.

EINFÜHRUNGSMATHEMATIK FÜR PHYSIKER, by Wilhelm Lenz. 95 pages, 21 × 15 cm. Germany, Wolfenbutteler Verlagsanstalt, Hannover, 1947. No price (paper).

This book is one of a series of "Notdrucke" which have recently appeared in Germany. As the aforementioned term indicates, these books are to serve, as well as possible, in place of textbooks during the present time of extreme scarcity.

In the short space of less than one hundred pages, Dr. Lenz has presented a remarkably concise and simple exposition of the basic mathematics needed by students of the physical sciences.

After an introductory chapter dealing with the basic concepts of function, limit, and infinite series, the author develops the differential calculus. In fifteen pages the derivative is defined and interpreted geometrically, the rules for differentiation are developed, the derivatives of the elementary functions are obtained, and the expansion of functions in Taylor series is treated. The third chapter, on the integral calculus, covers a surprisingly large amount of material. In addition to the definition of the integral, methods for working out the most common types, and the interpretation in terms of area, there are brief but highly informative sections on Fourier series, Fourier integrals, and analytic functions of a complex variable. The concluding chapter presents a simple but extremely useful survey of the subject of ordinary and partial differential equations.

While the style is (unavoidably) very concise, resulting in a rather unrigorous development, this little book should prove extremely useful both to students attending courses and to those who wish to refresh their knowledge; on the other hand, it is definitely not to be recommended for self-teaching.

The reviewer knows of no comparable book in the English language. A translation, or at least a comparable work, might be of aid to many persons who wish to review their mathematics after a lapse of years.

BERNARD EPSTEIN.

VACUUM TUBE AMPLIFIERS, edited by George E. Valley, Jr. and Henry Wallman. 743 pages, 16 × 23 cm., tables, drawings and illustrations. New York, McGraw-Hill Book Co., 1948. Price, $10.00.

This eighteenth volume of the Radiation Laboratory Series brings to electronic design engineers the best of the theory and procedures for the design and construction of vacuum tube

amplifiers as developed at the M.I.T. Radiation Laboratories during the past war. Here in a rather lengthy book written by thirteen engineers, physicists, and mathematicians, is an account of the principles of linear circuit design as applied to amplifiers having certain extreme characteristics, such as: high fidelity, bandwidth, sensitivity, linearity, low drift, etc. The usual rule-of-thumb or brute force methods utilized in the design of the ordinary audio amplifier, with which we are most familiar, cannot be applied successfully to the circuits described in this book, especially where ultimate performance is desired in any of the characteristics noted above. Since most of the war-time work of these laboratories was in the fields of radar and allied arts, it is natural for the amplifiers considered here to emphasize the passage of transient signals. The theoretical basis for the reproduction of high fidelity transient signals is approached through the use of Laplace and Fourier transformers and their inverses. Examples of the use of the Laplace method to the cases of an n-stage RC-coupled pulse amplifier and to a linear-phase coupling network illustrate the ease of application of this mathematical tool. The procedure for obtaining the transient response of networks is presented as a guide to those readers who are concerned with application of method with a minimum of theory.

The theory is applied to the case of high fidelity amplifiers for direct and video pulses. Many most practical considerations are presented, including the gain/rise ratio, overshoot, and overshoot-oscillation duration for many special circuits. Pulses of widely varying magnitude require special amplifier design, for in this case the voltage ratio between the largest and smallest signals may be of the order of magnitude of 10^8. The high gain and large dynamic range impose special features which are different from those encountered in the use of high frequency peaking circuits used in high fidelity pulse amplifiers. Six amplifiers, representing a good cross section of design are presented in detail with an analysis of their design and operation.

The sections devoted to the theoretical and practical design of amplifiers for use in the range from direct current up to 200 mc. per second include consideration of stagger tuned circuits, band pass types, and feedback amplifiers for low and high frequency applications, as well as outlines for the measurement and testing of amplifiers.

Perhaps in no other commonly encountered field of application as radar does the engineer meet with the difficult problems of minimum detectable signal. The detectability of weak signals involves many aspects, not the least of which is the problem of amplifier sensitivity and noise. These two subjects are treated in a most rigorous manner, approximately one third of the book being devoted to these problems. As is done in the rest of the book, a long chapter on basic theoretical analysis is followed by chapters describing practical methods by which amplifiers can be designed and adjusted to detect the weakest possible signals through the use of minimal noise circuits.

Encompassing so much of the authors' past experience with their associated theoretical work, the book is definitely a real contribution to the electronic art. It will probably find its way into the personal libraries of many enginers in this field.

S. CHARP.

ROTARY VALVE ENGINES, by Marcus C. Inman Hunter. 216 pages, 14 × 21 cm., illustrations, tables and drawings. New York, John Wiley & Sons, Inc., 1946. Price, $5.00.

For over sixty years engineers and designers have sought to develop rotary valves as the ideal solution to many of the problems encountered in the design of internal combustion and steam engines. In spite of all the engineering thought and experimental work expended on their development, relatively few of these have remained in production and use, notwithstanding all the advantages claimed or realized. The author of this book traces the history of the development of the rotary valve and gives some of the possible sources of failure of so many of these promising designs.

Some of the important advantages claimed for this type of rotary valves in comparison with the conventional poppet-valves in an internal-combustion engine and the slide valve in a steam engine, are found in quiet operation, freedom from vibration, and the elimination of

inertia troubles. The reduction in the clattering of a number of poppet-valves in an automobile engine by rotary valves is in itself an important advantage, so long as other fundamental deficiencies are not allowed to outweigh it.

The general principles of rotary valve design are discussed and many drawings are used to illustrate all points. Naturally the greatest single source of failure of the various designs that have been tried in the past is the presence of excessive friction between the moving sleeve, disc or cylinder and the inlet or exhaust surfaces. This is the source of wear which in turn develops excessive clearances and, in addition, causes increased drag on the mechanism that ultimately affects the entire operation of the engine. The author shows by example how to overcome these various disadvantages by proper design.

One of the primary objects of this book is to bring together all of the available information on early rotary valve engines and to discuss the relative merits of each design. In this way the causes of past failures can be pointed out and avoided for any new designs. The advantages of various methods of overcoming most of the design difficulties are pointed out and the technical requirements are given to explain the reason for their adoption. There appears to be no question in the author's mind that the rotary valve engine is feasible and one of the best engineering solutions to the optimum design of internal combustion and steam engines.

This book gives very little mathematics on the subject, since this can always be found in other places, restricting itself to the design and its historical development.

F. R. SIMPSON.

MICROWAVE MIXERS, by Robert V. Pound. 381 pages, 16 × 23 cm., tables, drawings. New York, McGraw-Hill Book Co., Inc., 1948. Price, $5.50.

In common with many other volumes in the M.I.T. Radiation Laboratory Series, this 16th volume is based upon this laboratory's war-time experience with microwave receivers specifically designed for radar. The particular brand of receiver study considered in the book is concerned with the design of components used in the microwave energy region with microwave techniques.

While as noted above, the background material is rather specialized, the information contained in the book will probably be of value to those interested in the design of microwave receivers for other applications.

The design of receivers for microwave use is so specialized that the conventional method of discussing receiver design as a unified system problem is not very successful, and emphasis must be given to the microwave frequency and low frequency problems separately. In one respect, the mixer is the heart of the microwave receiver, since very broadly it is the device which yields, at its output terminals, a signal containing the same phase and amplitude relationships among its components as those impressed upon its input terminals, but with a center frequency differing from that of the input signal. The particular form of the mixer, however, depends upon a number of factors, such as required bandwidth, frequency, automatic frequency control requirements, noise limits, etc.

Fifteen types of microwave receivers are briefly described, and the numerous sketches throughout the book help the reader to understand the fundamental microwave techniques. In the greater part of the book, however, emphasis is placed upon the crystal rectifier unit as a detector of microwave signals. While the subject of crystal rectifier design is not treated in detail, there is an excellent rudimentary discussion of the physical mechanism of the crystal detector and of the linear network treatment of frequency conversion using a crystal mixer. The major problem considered in the network analysis is that of the impedance and/or admittance of the input and output circuits of a crystal mixer. Relationships between the two are developed, and limiting conditions are set forth whereby the expression for impedance loss may be calculated for measurements of the i-f admittance for short circuited and open circuited signal terminals. Other problems consider the dependence of r-f matching conditions, conversion loss and image reflection, and the effects of reflection of the second harmonic.

Considerations in the design of crystal mixers that perform, with the aid of the local oscillator, the function of frequency conversion of a microwave signal are considered in great

detail. The examples cited are those of designs developed at the Radiation Laboratory, and because circuits employing crystal units have to be "tailormade" to fit a particular purpose, the examples are of rather limited interest. Emphasis is placed, however, on general methods of analysis so that the design engineer may make analagous attacks when other frequency bands and uses are encountered.

The design and operation of mixer circuits, especially where used with cavities and wave guides, has led to new problems in closely allied fields. Multi-channel mixing, directional couplers, local oscillator noise, balanced mixers such as the magic-T, and frequency control of the local oscillator all present special problems of design which are considered by the author. "Microwave Mixers" is an excellent work for use by design and research engineers concerned with microwave receivers. Frequent reference to this text should save many hours of effort in the solution of problems in this field.

S. CHARP.

SUPERSONICS—THE SCIENCE OF INAUDIBLE SOUNDS, by Robert Williams Wood. 162 pages, 13 × 20 cm., drawings and illustrations. Brown University, Providence, Rhode Island, 1939. Price, $2.00.

Prior to the advent of World War II, comparatively little interest was shown by scientists in the field of supersonics. However, the past few years have shown a remarkable advance in the practical applications of this field and the interest in it has accelerated enormously.

Professor Wood, in writing this book, has performed a valuable service to those men now engaged in this field, and to all scientists who wish to become enlightened on the subject. The book, originally published in 1939, is actually a series of three lectures presented as the 1937 Clover Lectures at Brown University.

The first lecture concerns itself with the history of supersonics before World War I. It discusses, without too much detail, the early investigations of inaudible sounds and includes the contributions of such early experimenters as Despretz, Rudolph Koenig, Galton, Edelmann, Altberg, Schultze, Lebedev, and Neklepajev. Professor Wood goes into more detail with the introduction of Hartmann's acoustic generator and the various piezo-electric sound generators in the form of quartz crystals. Of particular interest are the experiments of Professor Langevin in Paris at the start of World War I on the use of quartz crystals to generate high frequency sound waves under water for the detection of submarines.

The second lecture covers the Tuxedo Park experiments which Professor Wood and Mr. A. L. Loomis conducted in 1927 to determine the physical and biological effects of supersonic waves. Included in this lecture are the results of other investigators on Transverse Waves, Effects of Supersonic Waves on Chemical Reactions and Crystallization, Luminous Resonators, Supersonic Wave Systems as Optical Gratings, and other interesting experiments.

In the third lecture emphasis is placed on the early experiments of Newton Gaines at the University of Texas with magneto-striction oscillators. There is also an interesting section on the work of Dr. Leslie Chambers using supersonics in medical research.

The book has now been reprinted with a supplementary bibliography, which is probably the most complete one ever published in this field. Although investigations in supersonics in the past fifteen years are not reported in this book, its value lies in the presentation of the history of supersonic research.

D. M. BIBERMAN.

DISSOCIATION ENERGIES AND SPECTRA OF DIATOMIC MOLECULES, by A. G. Gaydon. 239 pages, 14 × 22 cm., tables, drawings and illustrations. New York, John Wiley & Sons, Inc., 1947. Price, $5.00.

This book was written because of the lack of a work dealing exclusively or completely with the determination of dissociation energies. Dissociation energies of diatomic molecules can be determined by one of several methods common to the research physicist or chemist, among them by thermo-chemical means and spectroscopic means. This book, written by a specialist in the field of spectroscopy, therefore approaches the problem from this point of view, although there are chapters on the determination of these energies by thermal methods and by con-

trolled electron impact. To some extent the book reports upon research the author conducted in the determination of the energies of dissociation of a number of diatomic molecules, among which are the important cases of CO, N_2, and NO.

The heat required to dissociate a gram molecule of diatomic gas at standard temperature to monatomic gases at the same temperature is the energy of interest. A direct measurement of this energy is not possible by either of the two methods noted above. The spectroscopist, in determining the energy of dissociation, endeavors first to determine the energy levels of a molecular species, and then from these to calculate the energy required to take a single molecule from its lowest possible energy state to the condition in which the constituent atoms can just be removed to an infinite distance apart. This method introduces certain small differences in the magnitude of the energy as computed by thermo-chemical means, but a conversion is possible from one value to the other by introducing certain assumptions.

In order to enable the reader to follow the main subject matter of the book without constant reference to other textbooks, the author presents in a condensed form the salient features of the theory of molecular spectra, with definitions of quantum numbers, coupling cases, and selection rules. It is felt that the sections devoted to this review are too brief, and while the fundamentals are presented, they are so sketchy that only one previously acquainted with the elements of atomic physics can appreciate this part of the book.

For most purposes, the dissociation energy is measured from the lowest real energy level, which will lie slightly above the minimum of the potential curve. These curves may be derived from spectroscopic observations, and methods are presented for plotting the energy curve from numerical data through the use of any of several general purpose functions which have been derived for this purpose, such as those due to Morse, Dunham, Hulbert-Hirschfelder, and Klein. The potential energy curve, however, does not tell the complete story.

In order to determine the energy of dissociation with normal atoms, the atomic products of the dissociation must be known. In many cases these products may be excited atoms or even ions. The Wigner-Witmer rules help determine what molecular states will arise from a combination of any two atomic states. This correlation is examined in detail for a number of various atomic and molecular states and the non-crossing rules are derived.

A number of other subjects are considered in detail, among which are: continuous spectra and convergence limits, and the dissociation energies of oxygen and the halogens; the Birge-Spover extrapolation with typical examples; the phenomena of predissociation; and the methods of energy determination by electron impact methods and thermal and thermo-chemical methods.

Researchers in this field may find particular interest in the collection of the numerical data for about 250 diatomic molecules which give the most probable values of the dissociation energies for which adequate data exist. In view of the uniqueness of the subject matter, the book should have wide acceptance as a reference text by advanced graduate students in physics and by research physicists and chemists who are daily concerned with dissociation energies.

S. CHARP.

COLLEGE PHYSICS, by Henry A. Perkins. Third edition, 786 pages, 16 × 23 cm., tables, drawings and illustrations. New York, Prentice-Hall, Inc., 1948. Price, $6.65.

The science of physics is so broad in its scope that, with our broadening horizons, it becomes increasingly difficult to give the student a complete description and explanation of all the laws of nature, beginning at the most fundamental ones and following their various ramifications through to the latest discoveries on the frontiers of science, and yet enclose these thoughts in a volume of reasonable size. To give sufficient clarity to these descriptions so that either the beginner or the advanced student will gain the maximum use of the work requires careful selection to include the correct amount of explanation without wasting valuable space. The book at hand has made an excellent attempt to satisfy these requirements and, after the nineteenth printing of the first two editions, this third edition, with its complete modernization, indicates that a considerable success should be attained.

Those aspects of nature as well as man-made machines that are sufficiently simple to be

explained mathematically in terms of fundamental laws are considered as the legitimate field of physics and treated in this book. These are grouped under five heads: mechanics, heat, sound, light, and electricity. The introduction, and the beginnings of each chapter, start with a clear and simple explanation of the subject that is both understandable to the beginner studying the subject and to the busy designer and engineer who wishes to verify a physical law applicable to his work without the loss of too much time.

Only elementary algebra and trigonometry are used to give a symbolic statement of fundamental laws without traditional proofs in the classical manner. The basic equations derived are sufficient to give the beginner a thorough understanding of each subject without too much assistance from the instructor. The ample use of figures and illustrations throughout the text helps greatly in speeding the complete understanding of each point. Free use is made of comparisons with similar phenomena in order to explain difficult points in the text.

The first part treats the most fundamental subject—mechanics—as motion, force, and energy, together with their relation to each other and to matter. Starting with basic measurements, fundamental statics, kinetics, and concepts of work, the student is given a clear description of the behavior of bodies in motion, explanations of the laws of hydrostatics, gas flow, fluid motion and capillarity. The second part discusses heat and the laws of thermodynamics relating to gases, liquids and solids. The third part treats of the increasingly important subjects of wave motion and sound, giving physical descriptions and mathematical explanations of the various phenomena. The fourth part discusses the various laws concerning light and its use. The fifth part concerns electricity and magnetism with their application to many problems encountered by the engineer. The last part covering corpuscular physics describes the new concepts of the inner nature of matter as investigated by the quantum theory. This carries the subject through electron emission and X-rays to atomic structure, nuclear physicis and radioactivity.

Although a large portion of each subject is discussed in detail in the text, ample references are given to supplement the text and many practical problems are presented to illustrate its applications. This book is written on the level of the undergraduate engineer and physicist, for whom it should prove to be a sound foundation on which to build a specialized knowledge of engineering or physics.

F. R. SIMPSON.

MICROWAVE DUPLEXERS, edited by Louis D. Smullin and Carol G. Montgomery. First edition, 437 pages, 16 × 23 cm., tables, drawings and illustrations. New York, McGraw-Hill Book Co., Inc., 1948. Price, $7.50.

One very important radar problem which arose during the past war was that of the economic utilization of available equipment. In the field of radar, one phase of this problem was the use of a single antenna for both receiving and transmitting of ultra-high frequencies.

This is the subject matter in the 14th volume of the M. I. T. Radiation Laboratory Series, reporting on certain of the war-time developments of special interest for pulsed transmission applications. The term duplexer refers to a device or circuit used for high speed switching. In this respect a duplexer can be considered to be a fast double-pole, double-throw switch with low loss. A complete duplexer consists of two gas-filled switching tubes connected in a microwave circuit with three terminal transmission lines, one each for the transmitter, receiver and antenna. The gas-filled switching tube is called a TR tube if its function is to disconnect the receiver from the antenna, or an ATR tube when the function is to disconnect the transmitter.

The designers of duplexers today, just as those of several years ago, are faced with two major problems: (*a*) gas-filled switching tubes must be developed in conjunction with the circuits with which they are to be used; (*b*) the tubes operate under widely varying conditions with little known regarding the physics of operation. In linear or low level operations the phenomenon is much the same as the usual gas-filled tube so commonly used, but when in high level operation, the tube phenomenon is a non-linear one, with much research and development yet to be done in this field.

One third of the book is devoted to the linear properties of TR and ATR tubes, that is, where operation is limited to power levels less than that required to initiate a discharge in the tube. Many line sketches and graphs are presented to show construction details and performance characteristics.

The second third of the volume is devoted to the subject of microwave gas discharges of the type occurring in TR and ATR tubes. Prior to 1941 there was almost no knowledge available on gas discharges at frequencies higher than 300 mc. per second. The material presented here is in a comparatively new domain of physics, where experimental work is reported in the 3000, 10,000, and 24,000 mc. per second bands. This work could be carried out only because of the concurrent development of new equipment by which some of the phenomena could be observed. One study is of the spike of energy which appears at the start of the leakage power cycle, that is, it is the energy transmitted to the receiver during the time interval between the beginning of the transmitter pulse and the formation of the steady state discharge across the gap of the TR tube. Until recently this spike of energy had never been observed, but once it had been seen, its presence was studied as both a linear and a non-linear phenomenon.

The circuits used to connect the gas-filled switches to other components are discussed in the remainder of the book. Emphasis is placed upon branched duplexing circuits and balanced circuits. The former class includes the common T-shaped junction where the major problem is to arrive at a design which minimizes the losses over a broad band of frequencies. The latter class involves the combination of two magic T's and two TR tubes. Measurement techniques to determine the important properties of duplexers are discussed in a closing chapter of the work.

Since the book reports primarily on war-time developments, the duplexer circuits described are special-purpose ones, but much can be learned from the fundamentals of design which are presented. Throughout the book the concepts of linear circuit theory are applied to waveguides and duplexer design. Because of the special nature of this very important component, good design is shown to be greatly dependent upon a knowledge in three fields: (*a*) linear microwave circuits of most highly developed types; (*b*) properties of electric discharge in gases at microwave frequencies and at low and high levels; (*c*) best practices and techniques used in the construction of microwave tubes. For this reason the book is recommended for experimentalists in electronic physics and radar circuit designers. It should serve as a good reference work in this field.

S. CHARP.

FOURIER TECHNIQUE IN X-RAY ORGANIC STRUCTURE ANALYSIS, by A. D. Booth. 103 pages, 14 × 22 cm., drawings and illustrations. Cambridge University Press; New York, The Macmillan Co., 1948. Price, $2.75.

The technique of organic crystal-structure determination by means of Fourier synthesis has progressed during the past twenty years from crude approximations in very simple compounds to accurate evaluations of complex substances. The modern structural crystallographer is able to proceed from a table of experimental Fourier coefficients to a complete and accurate set of atomic parameters, although it must be admitted that this can be done only for the simplest molecules at present. This book attempts to give a comprehensive account of the mathematical and experimental techniques by which the various methods used in the past have attempted to determine crystal structure.

The treatment proceeds from fundamentals, briefly outlined and assuming an adequate knowledge of geometrical crystallography and the theory of space groups, to the more complex phases of Fourier synthesis. The effect of the interaction of X-rays on a finite crystal through radiation, the effect of electron distribution in the atom, the unit cell composition and the arrangement of the unit cells into finite crystals are the basic stages from which the development proceeds. These have as their ultimate object the determination of the structure factor and phase angle values, either by mathematical methods or by indications deduced from the various physical properties.

The basic method of approach is through a representation of the electron distribution by a Fourier series to determine the calculated structure factor. Various methods of mapping

of projected sections in two dimensions are used to obtain a three-dimensional synthesis of the crystal structure through determination of atomic coordinates and the location of atomic positions for determining the observed structure factor. It is difficult to obtain the phase angle with any degree of certainty.

Methods available or proposed for computation of the structure factor and Fourier synthesis are described, together with machines designed to do this work, ranging from simple, hand-operated ones to those involving use of the more complicated electronic and optical systems.

This book is for the reader who has an adequate knowledge of X-ray diffraction crystallography and higher mathematics. The complexity of the problem is so great that one is not surprised when the author admits that there is still an unsolved barrier to the extension of the single-crystal Fourier method to the complex domain of virus and protein structures in the form of an inadequate knowledge of the phase angle. With an intelligent and sincere study of the problem as outlined in this book, there is no reason why the problem cannot be solved ultimately.

F. R. SIMPSON.

FUNDAMENTALS OF PHYSICAL SCIENCE, by Konrad Bates Krauskopf. Second edition, 676 pages, 15 × 23 cm., tables, drawings and illustrations. New York, McGraw-Hill Book Co., 1948. Price, $4.50.

The author, who is Associate Professor of Geology at Stanford University, has in this book contributed a well-written text for the non-technical college student who is interested in acquainting himself with the fundamentals of physical science. As Professor Krauskopf himself states it, the object of the book is "to understand the world of which we are a part, to read the pages of the earth's long history, and to learn how man's ideas about the universe have developed." The project undertaken by the author is indeed a worthy one; and, although there may still be a few questions left unanswered in some students' minds, the book is a good elementary text.

In general, the materials for the book are taken from the four sciences of astronomy, physics, chemistry and geology. These broad fields are cleverly interwoven without resorting to complex mathematical formulae so that any intelligent college level student could easily understand it.

The book is divided into six major parts each of which is subdivided into chapters on related subjects. At the end of each chapter, the author has prepared problems for the students to consider.

Part I concerns itself with the solar system and serves as an introduction to the basic concepts of our universe, including the fundamental laws of force, motion, and gravitation. In Part II Professor Krauskopf discusses matter and energy and brings in the various states of matter, the kinetic theory, chemical change, the atomic theory, and the periodic law. Part III is devoted to the structure of matter and in this part the student is introduced to the laws of electricity and magnetism, light waves, X-rays, and the principles of radiation. In Part IV the author covers the fundamental processes which include the theory behind chemical reactions, and colloids. Part V is a biography of the earth and includes geological phenomena, weather and climate, rock studies, and the earth's history. The last part, Stars and Galaxies, concerns itself with the universe beyond the earth and includes chapters on the sun, the stars, and the nebulae.

The first edition of the book was published in 1941, and the second edition is now being released with a few changes to bring the text up-to-date in several rapidly advancing fields of science.

D. M. BIBERMAN.

ELECTRONIC INSTRUMENTS, edited by Ivan A. Greenwood, Jr., J. Vance Holdam, Jr., and Duncan MacRae, Jr. 721 pages, 16 × 23 cm., tables, drawings and illustrations. New York, McGraw-Hill Book Co., Inc., 1948. Price, $9.00.

The editors' announced intention to prepare a handbook for designers of "some" electronic instruments is realized in the cases of electronic analogue computers, instrument servomechanisms, and voltage and current regulators. The treatment of systematic design procedures, in addition to basic principles, specific circuits and circuit components, makes it possible for new workers in the field to proceed along a broad front. Particular instruments developed at the Radiation Laboratory during World War II are treated primarily as examples of what can be done by a careful worker.

The first seven chapters are devoted to the design of electronic analogue computers, so called because electrical networks are used to represent simple or complicated physical systems and to obey equations of the same sort. Many usable circuits and components are described for performing the operations of arithmetic and calculus and for generating non-linear functions; and methods are outlined by which large and complicated systems can be built.

The next seven chapters cover the theory, design and testing of low-power, light-weight servomechanisms, which perform several useful functions in analogue computers. The development of the art, for both linear and non-linear relations, is described adequately. An understanding of the sections on transient analysis and frequency analysis and a proper application of techniques for correction of performance are necessary aids to the achievement of specifications.

The design of voltage and current regulators for the power supplies of electronic instruments is described in two chapters, which include the characteristics of regulator elements and examples of practical regulator design.

The two chapters on pulse test equipment are not appropriate to the low frequency setting of the majority of the volume, yet, if the ASA definition of instruments is accepted, the radar test oscilloscopes described therein are the only examples of instruments in the volume.

A final chapter on the design and construction of electronic apparatus emphasizes the importance of systematic procedures, including a check list of design factors, and describes what kinds of circuit design and construction techniques are appropriate to laboratory equipment on the one hand and to commercial or military equipment on the other hand.

W. W. FELTON.

BOOK NOTES.

CLIMATIC ACCIDENTS IN LANDSCAPE-MAKING, by C. A. Cotton. 354 pages, 14 × 22 cm., drawings and illustrations. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd.; no date. Price, $7.00.

The author, a New Zeland geologist of note, has written this volume on climatic accidents to give them the attention they deserve. These interruptions in landscape making have profound influence on later formations and on the interpretations of those formations in later periods. Two major types of climatic landscape accidents have been considered. These are "Dry and Dry-Seasonal Climatic Landscape Types" and "Glaciated Landscapes." The treatment is detailed with many illustrations, both line cuts and plates, depicting the numerous types of resulting landscapes.

G. E. P.

HANDBOOKS OF TEXTILE TECHNOLOGY NO. 3, THE IDENTIFICATION OF TEXTILE MATERIALS. 10 pages, 15 × 24 cm., illustrations and plates. Manchester, The Textile Institute, no date. Price, 5s (Paper).

The Textile Institute, through its Unification of Testing Methods Committee, has prepared this tentative specification on the identification of textile materials. The schemes of tests to be used and the results obtained are presented in the form of tables, clearly indicating the rec-

ommended order of testing. Twenty-five photomicrographs are included illustrating the different fibers both longitudinally and in cross section.

G. E. P.

CONVERSION OF PETROLEUM, by A. N. Sachanen. Second edition, 602 pages, 15 × 23 cm., tables and drawings. New York, Reinhold Publishing Corp., 1948. Price, $11.00.

The marked progress in manufacturing motor fuels since the first edition of this book eight years ago has taken two forms. One is great increase in capacity—of catalytic cracking from 100,000 to 1,000,000 barrels of charging stock a day and of alkylation from 10,000 to 130,000 barrels of alkylate daily. The second change has been in the commercialization of other processes. These changes have necessitated a thorough revision of the book. This has been accomplished by the inclusion of two new chapters on "Conversion of Petroleum Gases into Motor Fuel" and "Catalytic Cracking," as well as the extensive rewriting of other chapters. All the bibliographies have been amplified and brought up-to-date.

G. E. P.

HISTORY OF FACTORY AND MINE HYGIENE, by Ludwig Teleky. 342 pages, 14 × 22 cm., tables. New York, Columbia University Press, 1948. Price, $4.50.

In this study on industrial hygiene Dr. Teleky throws light on the progress that has been made from ancient times to the Second World War. The survey covers the hygiene of the factory and its installations as well as of the work itself. Poisons, dust, skin diseases are treated with a special chapter on mining problems. Attention is paid to the growth of control measures and factory inspection in Germany, England and the United States, the three principal industrialized countries and those primarily considered in this study. This book is important as a record of what has been done and also as a suggestion of what still needs to be done to secure comparable and adequate industrial hygiene conditions in all countries. A bibliography is included.

G. E. P.

THE CHEMICAL CONSTITUTION OF NATURAL FATS, by T. P. Hilditch. Second edition, 554 pages, 15 × 25 cm., tables and drawings. New York, John Wiley & Sons, Inc., 1947. Price, $9.00.

This important treatise on the natural fats has been expanded by more than a hundred pages in this revised edition. The same general treatment is followed with consideration given first to the component acids of aquatic flora and fauna, land animals, and vegetables. The component glycerides of the same fats are then discussed. Other chapters are devoted to the biochemistry of fats, constitution of individual natural fatty acids, and synthetic glycerides. A concluding chapter offers notes on experimental technique employed in the quantitative investigation of fats.

G. E. P.

SYMPOSIUM ON SEARCHLIGHTS, 164 pages, illustrations and diagrams, 16 × 25 cm. London, Illuminating Engineering Society, 1948. Price, 17s (Paper).

This volume includes five papers presented at a meeting of the Illuminating Engineering Society, April 15, 1947, with discussion and one additional paper. "The function and design of Army searchlights" treats of developments since the first World War. "High-power searchlights capable of wide divergence" discusses an airborne searchlight for fast fighter aircraft and a high current ground searchlight. "Photometry of searchlights" considers new apparatus and techniques which have been devised to facilitate the study of searchlight performance. "The visibility of targets in a Naval searchlight beam" and "Some visibility problems associated with anti-aircraft searchlight beams" present results of investigations into visibility under varying conditions. The concluding paper is on "Aircraft searchlights for anti-submarine warfare."

G. E. P.

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION.

Abstract of **Reaction of Borate with Polysaccharides: Blood Group Substance from Intestinal Mucosa and Gastric Mucin.**—CHARLES A. ZITTLE, LOUIS DESPAIN SMITH AND LAURA E. KREJCI (*Archives of Biochemistry*, 19: 9, 1948). A polysaccharide related to the specific blood group substances was isolated from a calf intestinal mucosa phosphoesterase preparation by precipitation with borate from a solution containing ammonium sulfate.

The borate precipitation method was used to isolate the polysaccharides from calf intestinal mucosa and from commercial hog gastric mucin. Examination of the calf mucosa polysaccharide in the Tiselius electrophoresis apparatus demonstrated but one component which exhibited anodic migration in buffer of pH 7.6, with a mobility of 3.3×10^{-5} cm.^{2} sec.$^{-1}$ volt^{-1}. The mobility was constant from pH 8.5 to pH 4.7; at pH 3.0 the mobility was 2.7×10^{-5}. The polysaccharide isolated from the hog mucin showed two components, the faster component having a mobility of 3.9×10^{-5} cm.^{2} sec.$^{-1}$ volt^{-1} and the slower 1.5×10^{-5}.

Serologically, the calf mucosa polysaccharide was found to be closely related to both Type A and Type B red cell substances, the Type A activity being about ten times greater. The hog mucin polysaccharide was several thousand times more active against Type A agglutinins than against Type B agglutinins.

The calf mucosa polysaccharide was found to contain 5.3 to 5.6 per cent nitrogen, and yielded 51 to 54 per cent reducing sugar on hydrolysis. The hog mucin polysaccharide was found to contain 4.6 per cent nitrogen and yielded 58 per cent reducing sugar on hydrolysis.

The interaction of borate and polysaccharide is believed to be an extension of the well-known interaction of borate and simple carbohydrates.

CURRENT TOPICS.

Seven Mile Conveyor Placed in Operation; Belt Vulcanized on Job (*Industry and Power*, Vol. 55, No. 5).—A seven mile long belt conveyor system, said to be the second largest ever constructed, was recently placed in operation to haul raw materials for construction to the Bull Shoals Dam on the White River near Flippin, Arkansas. The extensive transport system will operate cross-country in a series of 21 flights from quarry to dam site. Built in sections, the 36-in. wide belting is being spliced together on the job by means of a portable electric vulcanizer.

The series of main haul belts weigh 360 tons, comprising 250,000 lb. of cotton fabric and 470,000 lb. of rubber. Belting was furnished by the Goodyear Tire and Rubber Co.

Machinery for the extensive conveyor was manufactured by Hewitt-Robins, Inc., and consists of 13,888 rollers, each made up of three tube-shaped steel pulleys. Conveyor runs at 525 ft. per minute and delivers 650 tons of rock per hour.

When completed in December, 1950, Bull Shoals Dam will be the country's fifth largest water storage structure. More than 4,000,000 tons of rock and 1,800,000 barrels of cement will be required to construct the 283-ft. high barrier. The dam will act as a flood control, will irrigate 1,000,000 acres of land, and provide power to satisfy the present requirements of the cities of Little Rock, Ark.; Tulsa, Okla.; and Springfield. Mo.

Spreader-Type Stoker Has Hydraulic Drive (*Industry and Power*, Vol. 55, No. 5).—Complete combustion of any grade of coal is possible with the recently announced Westinghouse spreader-type industrial stoker. Coal is trajected by an air cooled, hydraulic motor-driven rotor and falls evenly over the entire length of the grate. Ashes can be discharged without interrupting the coal feed, reducing load, or loss of steam pressure. Coal is fed to the rotor by two long-stroke, slow-speed rams. A coal-feed equalizer plate driven from the rotor oscillates across the incoming fuel to assure a continuous feed whether the coal is wet or dry. One constant speed motor or turbine supplies oil pressure for all controls and drives. This hydraulic drive permits maintaining accurately any preset fuel-air ratio.

New "Snap Nuts" (*Radio & Television News*, Vol. 40, No. 5).—Prestole Corporation of Toledo is currently introducing a new spring steel, self-locking "Snap Nut" which is said to expedite the anchoring of nut-to-panel for blind attachments.

With the new unit no welding, riveting, clinching, or special tools are necessary. The nut is simply pressed into assembly position. It is attached directly from the work surface instead of the reverse side of the panel as is commonly done. It snaps into a 9/32 square hole in panels .037 to .005 in. thick and is designed for easy entrance of the screw even in cases of extreme misalignment of panels.

As the screw is driven, the arched spring arms of the nut expand just enough to permit entry of the screw, locking the fastener to the inner panel, and at the same time binding against the root of the screw thread. This new fastener will accomodate both No. 8 and No. 10 sheet metal screws, the larger size merely expanding the spring arms. The unit will withstand a tightening torque of 35 to 45 inch pounds.

According to the company, the nut is particularly suited for blind attachments on radios, automobiles, refrigerators, and other mass production products.

Atomic Energy Is at Work for Us (Excerpts from remarks made by Commissioner Sumner T. Pike, at the opening of the Atomic Energy Exhibit at The Franklin Institute.)—". . . I should like to quote a few words from the Report of the President's Committee on Higher Education which was made public early this year: 'The release of atomic energy that has brought man within sight of world devastation has just as truly brought him the promise of a brighter future. The potentialities of atomic power are as great for human betterment as for human annihilation. Man can choose which he will have. The possibility of this choice is the supreme fact of our day, and it will necessarily influence the ordering of educational priorities. We have a big job of re-education to do. Nothing less than a complete reorientation of our thinking will suffice if mankind is to survive and move on to higher levels.'

". . . You as citizens will be called on to make your opinions known and to cast your votes as this great project continues to grow. It isn't necessary to be a technically competent scientist in order to have worthwhile opinions on these developments any more than it was to be an automobile mechanic, an electronics engineer, or an aviator, to make up your mind about things which require public handling in those fields. What we need is an understanding with a few simple facts, some of which I have touched on briefly, and a feeling that this nation will somehow bring these developments into its accepted pattern of operation. This is a peculiar animal and it will take a lot of debate and experience before we can find out how best to hitch it onto our kind of wagon.

"The destructive and constructive applications of atomic energy are so closely involved with one another that it is going to be difficult for some time to preserve the proper balance between the necessities for government control and the widest applications in American industry. This we can do, I am certain, because we have met problems like these before. To do so requires patience, a sense of balance, a feeling for preservation of what is best in our system of society. It requires the kind of judgment this nation has often demonstrated. And it is public educational forums such as provided by this exhibit and this meeting that help prepare us for the task of making the fateful decisions that lie ahead of us."

Nylon-Coated Wire Rope (*Plastics*, Vol. 8, No. 11).—A nylon wire rope called "Wirelon" is now in commercial production at Rochester Ropes, Inc., Culpeper, Virginia. This new nylon-coated cable promises to open up extensive new-product possibilities as well as actually to lengthen the life and increase the usefulness of steel cables in many services where wire rope is already used.

Although chemically similar to the nylon used in women's hose, the coating

on Wirelon is not woven or filamented, but solid. It is almost incredibly tough.

Product-wise, the possibilities of Rochester Wirelon include automotive brake and speedometer cable, aircraft control cables, sash cords, agricultural and earth-moving machinery, fishing lines, and others running from grain cleaning equipment to slim, crystal-like dog leashes and swank leather handles. Wirelon will be available in diameters ranging from 1/32 to 1 in. or greater.

The Du Pont nylon coating is unaffected by oils, cleansing agents, marine growths, and all alkalies. It is not harmed by such acids as nitric, hydrochloric, or sulphuric in concentrations up to approximately 1 per cent. Temperature tests show that Wirelon ropes operate well at high and low temperatures. The ropes retain their flexibility whether wet or dry.

The coating is being used by Rochester in the manufacture of three distinct types of rope. In the first, nylon is extruded on to single strands of wire, which can then be used in many services where more expensive fully formed ropes were heretofore required. The second type is more flexible and is made by applying the plastic to the outside of wire ropes that are already completely formed. Still more flexible than either of the other types, is nylon-wire rope made by the third method, in which individual strands are coated with nylon and the coated strands then closed into a finished rope.

Electronic Pencil Enables Composers to Hear Score (*Science News Letter*, Vol. 54, No. 20).—Composers may now play the music they have written on a simple electrical machine by writing out the score with an electronic pencil.

The new composer's music box was described in Cleveland to the Acoustical Society of America by Prof. Robert B. Watson of the University of Texas physics department and defense research laboratory.

"Some composers are able to hear the music mentally at the time of composition," Professor Watson explained, "but others desire to hear certain musical passages while composing. A composer may now use a compact electro-acoustic device to produce various tones with comparative ease and little distraction from writing his musical score."

The music box is operated similarly to writing with paper and pencil. The tone is started by placing the electronic pencil on the plate of the instrument and stopped by lifting it.

Pitch is increased by moving the pencil upward. Loudness is increased by moving the pencil to the right. Tonal qualities of the sound can be obtained by varying the circuits in the machine. Skill of the player enters in variations of timing, pitch and volume.

Oxygen Enrichment Speeds Steel Production (*Discovery*, Vol. IX, No. 10).—Steel that, less than a year ago, took 15 minutes to make is now made in 8 at a steel foundry in Leeds, thanks to the efforts of the industry's scientists. Staff at the works concerned, cooperating with scientists from the British Iron and Steel Research Association have added extra oxygen to the air blast in side-blown Bessemer Converters to make a 50 per cent increase in output possible. Wear on the furnaces is no greater, and the high quality of the steel is unaffected. Fuel and pig iron are saved since the molten iron "charge" can be cooler to start with, and can contain more scrap. All over the world similar efforts are being made to bring down to works practice the theoretical advantages of oxygen enrichment, but this is believed to be its first use in regular production routine.